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The strong ABC conjecture over function fields
[after McQuillan and Yamanoi]

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1 Introduction.

One of the deepest conjecture in arithmetic is the *abc* conjecture:

1.1 Conjecture. *Let $\epsilon > 0$, then there exists a constant $C(\epsilon)$ for which the following holds: Let a, b and c three integral numbers such that $(a, b) = 1$ and $a + b = c$ then*

$$\max\{|a|, |b|, |c|\} \leq C(\epsilon) \left(\prod_{p/abc} p \right)^{1+\epsilon}$$

where the product is taken over all the prime numbers dividing abc.

Let's give a geometric interpretation of this conjecture:

Consider the arithmetic surface $\mathbb{P}^1_{\mathbb{Z}} \to \mathrm{Spec}(\mathbb{Z})$ equipped with the tautological line bundle $\mathcal{O}(1)$ and the divisor $D := [0 : 1] + [1 : 0] + [1 : -1]$. Suppose we have a section $P : \mathrm{Spec}(\mathbb{Z}) \to \mathbb{P}^1_{\mathbb{Z}}$, not contained in D, then $P^*(D)$ is an effective Weil divisor on $\mathrm{Spec}(\mathbb{Z})$ which can be written as $\sum_p v_p(D)[p]$.

Define *the radical* of the divisor as $N^{(1)}_D(P) := \sum_p \min(1, v_p(D)) \log(p)$.

The conjecture can be stated in this way: for every $\epsilon > 0$ there is a constant $C(\epsilon)$ such that, for every section $P : \mathrm{Spec}(\mathbb{Z}) \to \mathbb{P}^1_{\mathbb{Z}}$ we have

$$h_{\mathcal{O}(1)}(P) \leq (1 + \epsilon) N^{(1)}_D(P) + C(\epsilon).$$

Where $h_{\mathcal{O}(1)}(P)$ is the height of P with respect to $\mathcal{O}_{\mathbb{P}^1}(1)$. When we state the conjecture in this way we see many possible generalizations. We also clearly see the geometric analogue over function fields (cf. next sections for details). Let's formulate the conjecture in the most general version.

If K is a number field, we denote by $\mathcal{O}_K$ the ring of integers of K and Δ_K its discriminant. If $X \to K$ is an arithmetic surface. D is an effective divisor over X and $P : \mathrm{Spec}(\mathcal{O}_K) \to X$, not contained in D, we define the *radical* of D as the real number $N^{(1)}_D(P) := \sum_{\mathfrak{p} \in \mathrm{Spec\,max}(\mathcal{O}_K)} \min\{1; v_{\mathfrak{p}} P^*(D)\} \log Card(\mathcal{O}_K/\mathfrak{p})$. The general strong *abc* conjecture is the following:

1.2 Conjecture. *Let $\epsilon > 0$, and K be a number field, $\pi : X \to \mathrm{Spec}(\mathcal{O}_K)$ a regular arithmetic surface and $D \hookrightarrow X$ an effective divisor on X. Denote by $K_{X/\mathcal{O}_K}$ the relative dualizing sheaf. Then there exists a constant $C := C(X, \epsilon, D)$ for which the following holds: let L be a finite extension of K and $P : \mathrm{Spec}(\mathcal{O}_L) \to X$, not contained in D, then*

$$h_{K_{X/\mathcal{O}_K}(D)}(P) \leq (1+\epsilon)(N_D^{(1)}(P) + \log|\Delta_L|) + C[L:K]$$

where $h_{K_{X/\mathcal{O}_K}(D)}(P)$ is the height of P with respect to $K_{X/\mathcal{O}_K}(D)$.

We will not list here the endless number of consequences of this conjecture and we refer to [BG] or to the web page [NI] for details. One may also see the report [OE] in this seminar. We only notice that, if such a conjecture was true, more or less all the possible problems about the arithmetic of algebraic curves over number fields would have an effective answer: for instance one easily sees that, if the constant C is effective, it easily implies the famous Fermat Last Conjecture (now a theorem [WI]) and it allows to solve effectively diophantine equations in two variables:

1.3 Theorem. *Suppose that conjecture 1.2 is true. Let $F(x;y) \in \mathbb{Z}[x,y]$ be an irreducible polynomial of degree at least three. Then there exists a constant C, depending only on F, such that for every number field K and for every solution $(x;y) \in \mathcal{O}_K \times \mathcal{O}_K$ of the diophantine equation $F(x;y) = 0$, we have*

$$h_{\mathcal{O}(1)}([x:y:1]) \leq (1+\epsilon)\log|\Delta_K| + C_\epsilon[K:\mathbb{Q}].$$

In particular there are only finitely many solutions in $\mathcal{O}_K \times \mathcal{O}_K$ and their height can be explicitly bounded.

Observe that, if the conjecture is true and the constant C_ϵ is explicit, then we can explicitly *compute and find* the set of solutions of the diophantine equation in $\mathcal{O}_K \times \mathcal{O}_K$.

Similarly we may obtain an effective version of Mordell conjecture (Faltings theorem) and of the classical Siegel theorem on integral points of hyperbolic curves.

At the moment we know that the set of integral points of an hyperbolic curve is finite (projective by Faltings theorem [FA] or affine by Siegel theorem cf. [SE]) but we are not able to explicitly bound their height (up to some sporadic cases); thus, in particular, it is not possible to find all the rational points of an hyperbolic curve.

In this paper we will report about the solution of the analogue of the *abc* conjecture over function fields (for the analogy between number fields and function fields arithmetic cf. for instance [SE]).

The analogue of conjecture 1.1 for polynomials is quite easy and proved in [MA]: If f is a polynomial over $\mathbb{C}$ (to simplify), let $N_0 = (f)$ be the number of *distinct* roots of f. Then the analogue of the *abc* conjecture for polynomials is

1.4 Theorem. *(Mason) Let f, g and h three polynomials relatively coprime in $\mathbb{C}[t]$*

such that $f+g=h$, *then*

$$\max\{\deg(f),\deg(g),\deg(h)\}\leq N_0(fgh)-1.$$

This theorem is the analogue of conjecture 1.2 for function fields when $X=\mathbb{P}^1\times\mathbb{P}^1$, $\pi: X\to\mathbb{P}^1$ is the first projection, $D=\mathbb{P}^1\times[0:1]+\mathbb{P}^1\times[1:0]+\mathbb{P}^1\times[1:-1]$ and P is a section. One easily deduce it from Hurwitz formula (cf. next section). It can be seen as the beginning of all the story, and it has some interesting consequences: for example the analogue of Fermat last theorem for polynomials is an immediate consequence of it. Usually statements in the function fields situation are much easier to prove then their correspondent in the number fields situation. In this case one should notice an amazing point: Suppose that, over number fields, we can prove conjecture 1.2 when $X=\mathbb{P}^1_{\mathbb{Z}}$ and $D=[0:1]+[1:0]+[1:-1]$ then we can deduce from this the general case! To prove this one applies the proof of theorem 7.1 to a suitable Belyi map (for more details cf. [EL]). In the function fields case this is *not* the case! We cannot deduce the general case from an isotrivial case. For this reason it is our opinion that $\mathbb{P}^1_{\mathbb{Z}}$ with the divisor $[0:1]+[1:0]+[1:-1]$ (unit equations) is a highly non isotrivial family over $\mathrm{Spec}(\mathbb{Z})$ (whatever an isotrivial family should be).

Exploiting the analogy between the arithmetic geometry over number fields and the theory of *analytic* maps from a parabolic curve to a surface (cf. for instance [VO1]), an analogue of the *abc* conjecture for these maps also is solved.

We will propose two proof of the *abc* conjecture over function fields (and for analytic maps). The first is the proof by McQuillan [MQ3] and the second is by Yamanoi [YA3]. The proof by McQuillan is synthetically explained in the original paper; it make a systematic use of the theory of integration on algebraic stacks; although this is very natural in this contest, it needs a very heavy background (which here is used only in a quite easy situation). Thus we preferred to propose a self contained proof which uses the (easier) theory of normal $\mathbb{Q}$–factorial varieties; the proof follows the main ideas of the original one. The proof by Yamanoi requires skillful combinatorial computations, well explained in the original paper, thus we preferred to sketch his proof in a special (but non trivial) case: the main ideas and tools are all used and we think that once one understand this case, it is easier to follow the proof in the general situation.

As before, as a consequence we find, for instance, a strong effective version of Mordell conjecture over function fields (in characteristic zero), for non isotrivial families of hyperbolic curves.

In the next section we will explain why the *abc* conjecture for isotrivial curves corresponds respectively to the Hurwitz formula in the geometric case and to the Nevanlinna Second Main theorem in the analytic case. Thus the *abc* conjecture may be seen as a non isotrivial version of these theorems.

There are at least two strategies to attack the Second Main Theorem of Nevanlinna theory. The first strategy uses tools from analytic and differential geometry, it is strictly related to the algebraic geometry of the Hurwitz formula and to the existence of particu-

lar singular metrics on suitable line bundles: it has been strongly generalized to analytic maps between equidimensional varieties by Griffiths, King and others in the 70's (cf. [GK]). The second strategy is via Ahlfors theory (cf. [AH]) it is much related to the algebraic and combinatorial topology of maps between surfaces; the version of the SMT one obtain int this way is weaker then the original one but also more subtle: one sees that one can perturb a little bit the divisor D without perturbing the statement (cf. §8). These two approaches correspond respectively to the two proposed proofs. The Proof by McQuillan is nearer to the first strategy while the Yamanoi's is more topological. One should notice that, while the first proof is predominantly of global nature and the second is essentially local, both meet the main difficulties in an argument which is localized around the singular points of the morphism $p : X \to B$. If the morphism p is relatively smooth, McQuillan's proof is much simpler. In a hypothetical relatively smooth case, Yamanoi approach reduces to the Ahlfors theory: you will observe that, unless you are in the isotrivial case, in the Yamanoi approach there is always bad reduction.

Both proofs holds for curves over function fields in one variable over $\mathbb{C}$ and both heavily use analytic and topological methods, specific of the complex topology. We should notice that the analogue of the *abc* conjecture, as stated before, over a function field with positive characteristic is false! (cf. [KI]).

1.5 *A short overview of the history of the abc conjecture..* The *abc* conjecture has a weak and a strong version (in the arithmetic case they are both unproven and very deep). Over function fields, the weak *abc* is easier to prove and it is strictly related with the theory of elliptic curves (cf. [HS] and [SZ]). Here we deal with the strong version. The conjecture have been formulated in the middle 80's by Masser and Oesterlé exploiting the analogy between number fields and function fields and the version for polynomials proved in [MA]. The general version, as stated here, have been formulated by Vojta in [VO1] as a consequence of a series of conjectures for varieties of arbitrary dimension. The particular case of $\mathbb{P}^1 \times \mathbb{P}^1$ and $D := [0:1] \times \mathbb{P}^1 + [1:0] \times \mathbb{P}^1 + [1:1] \times \mathbb{P}^1 + \Delta$ (Δ being the diagonal) was previously proposed by Oesterlé. Some weak versions of the conjecture in the contest of the Value distribution theory have been proved in the papers [SA] and [OS].

In the paper [VO2] one find a proof of a weak version of the conjecture (with factor $2+\epsilon$ instead of $1+\epsilon$) in the algebraic case when D is empty (function field case!); it can be easily generalized to the case when D is arbitrary. In the recent paper [MY] we can find an algebraic proof of a weak version of the conjecture. On the preprint [CH] one can find another overview of the proofs.

Recently one can find generalizations of the theory in the papers [YA1] and [NWY]. A strong generalization (and many other results), for families of surfaces, is proved in the forthcoming book [MQ4].

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2 Notations and overview of Value Distribution Theory.

If X is a set and $U \subseteq X$ is a subset, then we denote by $\mathbb{I}_U$ the characteristic function of U.

Let X be a smooth variety defined over $\mathbb{C}$ and L a line bundle D a Cartier divisor over X such that $L = \mathcal{O}_X(D)$. We Suppose that L is equipped with a smooth metric.

Over X we have the two operators $d := \partial + \overline{\partial}$ and $d^c := \frac{1}{4\pi\sqrt{-1}}(\partial - \overline{\partial})$.

The divisor D corresponds to a section $s \in H^0(X, L)$ (defined up to a non zero scalar). The Poincaré–Lelong equation is

$$dd^c \log \|s\|^2 = \delta_D - c_1(L)$$

where δ_D is the Dirac distribution "integration on D" and $c_1(L)$ is the first Chern form associated to the hermitian line bundle L.

A divisor on X is said to be *simple normal crossing* (snc for short) if $D = \sum D_i$ with D_i smooth and locally for the Euclidean topology, we can find coordinates $x_1, \ldots, x_n$ on X for which $D_i = \{x_i = 0\}$ and $D = \{x_1 \cdots x_r = 0\}$. If D is snc, we can introduce the sheaf of differentials on X with logarithmic poles along D: $\Omega^1_X(\log(D))$; this is the sheaf of meromorphic differentials ω which may locally be written as $\omega = \sum_{i=1}^r a_i \frac{dx_i}{x_i} + \alpha$ where a_i are C^∞ functions and α is a smooth differential. If D is a divisor on a variety, we let D_{red} be the *reduced* divisor having the same support as D.

Suppose that Y is a *compact* Riemann surface and $f : Y \to X$ is an analytic map such that $f(Y) \not\subset D$. Since Y is without boundary, Stokes theorem gives

$$\deg(f^*(D)) = \int_Y f^*(c_1(L)).$$

Thus *the degree* of the restriction of the divisor D to Y can be interpreted as the *area* of Y with respect to the measure defined with $c_1(L)$. We will denote by $N_D^{(1)}(Y)$ the degree of $f^*(D)_{red}$; observe that $N_D^{(1)}(Y) = \sum_{z \in Y} \min\{1, v_z(f^*(D))\}$ thus, in the number fields–function fields analogy, it corresponds to the radical defined in the previous section. In the sequel we will denote by (L, Y) the integral number $\deg(f^*(L))$ (omitting the reference to f if this is clear from the contest).

Suppose that $X = X_1 \times X_2$ where X_i are compact Riemann surfaces and $p_i : X \to X_i$ are the projections. Suppose that D_2 is a reduced divisor on X_2 and $D := p_2^*(D_2)$. We consider $(X; D)$ as an isotrivial family of curves with divisors over X_2 via p_2.

Let Y be a compact Riemann surface with an analytic map $f : Y \to X$. Call $f_i := p_i \circ f$. For $i = 1, 2$, define R_{f_i} as the Ramification term: $R_{f_i} := \sum_{z \in Y}(Ram(f_i) - 1)$. The Hurwitz formula gives $f_2^*(\Omega^1_{X_2}(\log(D_2))) \hookrightarrow \Omega^1_Y(f^*(D)_{red})$. Thus a double application

of the Hurwitz formula gives

$$\deg(f^*(K_{X/X_1}(D))) \leq N_D^{(1)}(Y) + R_{f_1} + \chi(X_1)\deg(f_1);$$

Which is the analogue (which holds with $\epsilon = 0$) of the *abc* conjecture over function fields in the isotrivial case.

When $f : Y \to X$ is an algebraic map between smooth projective curves, we will denote by $[Y : X]$ the degree of the pull back, via f, of a generic point; it coincides with the degree of the field extension $\mathbb{C}(Y)/\mathbb{C}(X)$.

Suppose that Y is not compact. In this case we suppose that Y is *parabolic* equipped with an exhaustion function g: an exhaustion function is a unbounded function g such that $dd^c(g) = \delta_S$ where $S = \sum P_i$ is a reduced divisor of finite degree and near each P_i we can find an harmonic function h_P such that $g = \log|z-P|^2 + h$. Remark that g is harmonic outside S,

Examples: a) $\mathbb{C}$ with the function $\log|z|^2$

b) If $\pi : Y \to \mathbb{C}$ is a proper map of degree $[Y : \mathbb{C}]$ (not ramified over 0), then $(Y; \pi^*(\log|z|^2))$ is parabolic: thus every affine Riemann surface is parabolic.

c) If Y is parabolic and E is a polar set in Y then $Y \setminus E$ is parabolic.

For more details on parabolic Riemann surfaces cf. [AS].

Parabolic Riemann surfaces are the ones where one can develop a value distribution theory. We fix a parabolic Riemann surface (Y, g).

Suppose that $f : Y \to X$ is an analytic map. We define the *intersection* of the hermitian line bundle L with Y as a *function* on $\mathbb{R}$:

$$(L;Y)(r) := \int_{-\infty}^{\log(r)} ds \int_{g\leq s} f^*(c_1(L)) = \int_0^r \frac{dt}{t}\int_{g\leq\log(t)} f^*(c_1(L))$$

(in value distribution theory this is denote as $T_f(L)$; we choose this notation to stress the analogy with intersection theory). The intersection $(L, Y)(r)$ can be seen as *the average of the areas of the disks* $g \leq s$ for $s \leq \log(r)$. Up to a constant $(L;Y)(r)$ do not depend on the choice of the metric on L.

We will define the *non integrated counting function* as $n_D(s) := \sum_{g(z)<s} v_z(f^*(D))$ and the *the non integrated radical function* as $n_D^{(1)}(s) := \sum_{g(z)<s} \inf\{1, v_z(f^*(D))\}$; the n.i. counting function measure the growth of the degree of the divisor $f^*(D)$ on the disk $g < s$ and the n.i. radical plays the role of the radical . We define the *integrated counting function* and the *integrated radical* as

$$N_D(Y)(r) := \int_{-\infty}^{\log(r)} n(f^*(D), s)ds \text{ and } N_D^{(1)}(Y)(r) := \int_{-\infty}^{\log(r)} n^{(1)}(f^*(D), s)ds \text{ resp.}$$

In the same way we define a *non integrated characteristic function or ramification term*: the form ∂g is holomorphic outside S thus we define $r_g(s) := \sum_{g(z)<s} v_z(\partial g)$, where the sum is extended to points not in S. For instance, if Y is a proper covering of $\mathbb{C}$, then $r_g(s)$ is the degree of the part of the ramification divisor of the covering

supported in $g < s$. Thus we define the *integrated characteristic function* as

$$\chi(Y)(r) := \int_{-\infty}^{\log(r)} r_g(s)ds.$$

Integrating the Poincaré–Lelong equation we obtain the *First Main Theorem* of Value distribution theory (cf. [NE] or [HA]): Suppose that X is smooth projective , D and L are as before and $f : Y \to X$ then we can find an explicit constant C, independent on r, such that

$$(L; Y)(r) = N_D(Y)(r) - \int_{g=r} \log \|s\|^2 d^c g + C;$$

The term $m_D(Y, r) := -\int_{g=r} \log \|s\|^2 d^c g$ is called *the proximity function* and measure the average of the inverse of the distance of the image of boundary of $g \leq r$ from D.

Suppose that $X := X_1 \times X_2$ and $D := p_2^*(D_2)$ as before. Suppose that $f : Y \to X$ is an analytic map from a parabolic Riemann surface. The *Second Main Theorem* of Value Distribution Theory ([NE] and [HA]) can be stated in this way:

$$(K_{X/X_1}(D); Y)(r) \leq N_D^{(1)}(Y)(r) + \chi(Y)(r) + O(\log(r(K_{X/X_1}(D); Y)(r)))$$

where the involved constant is independent on r. Thus, the analogue of the *abc* conjecture in the isotrivial case is the second main theorem.

Let Y be parabolic, B be a compact Riemann surface and $f : Y \to B$ be an analytic map. Let $P \in B$ be a point and equip $\mathcal{O}_B(P)$ with a smooth metric. In analogy with the algebraic case, we will denote $[Y : B](r)$ the function $(\mathcal{O}_B(P); Y)(r)$.

3 Statement of the main theorems.

In this section we will state the main theorems, namely the *abc*–conjecture over function fields and make the first easy reductions.

The object of study is a set (X, D, B, p) where, X is a smooth projective surface, D is a simple normal crossing divisor on X, B is a smooth projective curve and $p : X \to B$ is a non constant morphism. We will also fix an ample line bundle H equipped with a smooth positive metric.

We will explain the proof of the two theorems below, one is in the algebraic and the other in the analytic setting. They correspond each other in the analogy and we will see that the proofs are, *mutatis mutandis*, very similar.

3.1 Theorem. (*abc* algebraic version) *Let $p : X \to B$ and D as above and $\epsilon > 0$. Then, given a smooth projective curve Y and a morphism $f : Y \to X$ whose image is not contained in D, the following inequality holds*

$$(K_X(D); Y) \leq (1 + \epsilon)(N_D^{(1)}(Y) + \chi(Y)) + O_\epsilon([Y : B]).$$

The involved implicit constants depend only on X, D, p and ϵ.

To avoid trivialities we supposed that the morphism $p \circ f : Y \to B$ is non constant.

3.2 Theorem. (abc analytic version) *Let $p : X \to B$ and D as above and $\epsilon > 0$. Let (Y, g) be a parabolic Riemann surface and $f : Y \to X$ be a holomorphic map with dense image. Then the following inequality holds*

$$(K_X(D); Y)(r) \leq (1+\epsilon)(N_D^{(1)}(Y)(r) + \chi(Y)(r)) + O_\epsilon([Y : B](r) + \log(r(H; Y)(r))). \quad //$$

The involved implicit constants depend only on X, D, p, f and ϵ but independent on r.

The symbol // means that the inequality holds outside a set of finite Lebesgue measure.

3.3 *Reductions and observations.* a) The theorems remain true if we change X by a blow up (but the involved constants may vary). Consequently they are statements about the algebraic curve X_K where K is the function field $\mathbb{C}(B)$. Moreover we may suppose that every irreducible component of D dominates B.

b) In order to prove the theorems we may take finite extensions of the base field $K := \mathbb{C}(B)$: if the theorem is true over a finite extension, it is true over it, and conversely. We may, and we will, suppose for instance that B is hyperbolic.

c) By the semistable reduction theorem, we may suppose that $K_X(D)$ is nef and big and that the fibres of p are reduced simple normal crossing. Incidentally, this shows that the hypothesis that D is simple normal crossing is unnecessary.

d) Suppose that $X = \mathbb{P}_1 \times B$ and D is the pull back, via the the first projection of the divisor $0 + 1 + \infty$; then the theorems give the "classical" *strong abc*–conjecture over function fields.

e) Suppose that, in the analytic situation, $(Y, g) = (\mathbb{C}, \log |z|^2)$, $X = \mathbb{P}_1 \times \mathbb{P}_1$, D is the pull back via the first projection of a divisor $\sum_{i=1}^d a_i$ and $f := (f_1, id) : \mathbb{C} \to X$, where f_1 is a meromorphic function, then the theorem becomes (in the standard notation of Nevanlinna theory)

$$(d-2)T_{f_1}(r) \leq (1+\epsilon) \sum_i N^{(1)}(a_i, f) + O_\epsilon(\log(rT_{f_1}(r))) \quad //.$$

which is essentially (up to the factor ϵ) the Nevanlinna Second Main Theorem.

4 The tautological inequality.

Let X be a smooth projective variety and $D \subset X$ be a simple normal crossing divisor. Let $\Omega^1_{X/k}(\log(D))$ be the sheaf of differentials of X with logarithmic poles along D and $\pi : \mathbb{P} := \mathrm{Proj}(\Omega^1_{X/k}(\log(D))) \to X$. We will denote by $\mathbb{L}$ the tautological line bundle on $\mathbb{P}$. We also fix an ample line bundle H on X

Let Y be a smooth projective curve and $f\colon Y \to X$ a map. The induced map f^* : $f^*(\Omega^1_{X/k}(\log(D))) \to \Omega^1_{Y/k}(f^*(D)_{red})$ induce a morphism $f'\colon Y \to \mathbb{P}$. By definition we have an inclusion $f'^*(\mathbb{L}) \hookrightarrow \Omega^1_{Y/k}(f^*(D)_{red})$ consequently we obtain the *tautological inequality*

$$(\mathbb{L}; Y) \le \chi(Y) + N^1_D(Y) \tag{4.1.1}$$

where $\chi(Y) := 2g(Y) - 2$ is the Euler characteristic of Y.

Suppose now that $k = \mathbb{C}$. We Suppose that H is equipped with a positive (Khäler) metric ω. We also equip $\mathcal{O}(D)$ and $\Omega^1_{X/k}(\log(D))$ with a smooth metric. Remark that the metric on $\Omega^1_{X/k}(\log(D))$ induces a metric on $\mathbb{L}$.

Suppose that $(Y; g)$ is a parabolic Riemann surface and $f\colon Y \to X$ is an analytic map whose image is not contained in D. The tautological inequality is the analogue of the 4.1.1 in this contest.

4.2 Theorem. *(Analytic tautological inequality) Let $f\colon Y \to X$ as above and $f'\colon Y \to \mathbb{P}$ the induced map. The following inequality holds*

$$(\mathbb{L}; Y)(r) \le N^1_D(Y)(r) + \chi(Y)(r) + O(\log(H; Y)(r)) \quad // \tag{4.2.1}$$

The tautological inequality above is an important push forward in the analogy between the algebraic and the analytic theory of maps of Riemann surfaces in projective varieties. It is very important because it translate the problems of defect type in Nevanlinna theory to problems of geometrical nature: If one prove that some intersection is upper bounded by the intersection with $\mathbb{L}$, one will deduce an inequality in the spirit of the Second Main Theorem of Nevanlinna theory.

4.3 Remark. One may wonder how much of the proofs of *abc* performed in the function field case can be done in the arithmetic situation. Unfortunately, in the arithmetic situation, the geometric interpretation of the radical via the tautological inequality is missing: one do not know what is the arithmetic meaning of the radical.

Proof: Write the divisor D as $\sum_i D_i$. Locally on X we can find coordinates $X_1, \dots, X_n$ in such a way that, there is an $r \in \{0, \dots n\}$ such that, each D_i is given by $\{X_i = 0\}$ and $D = \{X_1 \cdot \ldots \cdot X_r = 0\}$. Since X is compact, can choose a positive constant such that, the singular $(1,1)$ form

$$\omega^{sm} := A\omega + \sum_i \frac{d\|D_i\| \wedge d^c\|D_i\|}{\|D_i\|^2}$$

induces a smooth hermitian metric on $T_{X/k}(-\log(D))$. We introduce the singular $(1,1)$ form

$$\tilde{\omega} := \omega + \sum_i \frac{d\|D_i\| \wedge d^c\|D_i\|}{\|D_i\|^2(\log(\|D_i\|))^2}.$$

The form $\tilde{\omega}$ induces a singular hermitian form on $T_{X/k}(-\log(D))$: if we write an element of $T_{X/k}(-\log(D))$ as $t = \sum_{i=1}^{r} a_i \frac{X_i\partial}{\partial X_i} + \sum_{j=r+1}^{n} b_j \frac{\partial}{\partial X_i}$ then $\tilde{\omega}(t,t)$ is comparable to $\sum_i |a_i X_i|^2 + \sum_j |b_j|^2 + \sum_i \frac{|a_i|^2}{\log|X_i|^2}$.

Let $\mathbb{P}'$ be the projective bundle $\mathrm{Proj}(\mathcal{O}_X \oplus \Omega^1_{X/k}(\log(D)))$ over X, and let $\mathbb{M}$ be the tautological line bundle over it. The surjection $\mathcal{O}_X \oplus \Omega^1_{X/k}(\log(D)) \to \Omega^1_{X/k}(\log(D))$ induces an inclusion $\mathbb{P} \hookrightarrow \mathbb{P}'$ and $\mathcal{O}_{\mathbb{P}'}(\mathbb{P}) = \mathbb{M}$. Observe that, locally on X, $\mathbb{P}'$ is isomorphic to $X \times \mathbb{P}_n$ and, we may choose homogeneous coordinates $[z_0 : \cdots : z_n]$ on $\mathbb{P}_n$ for wich the divisor $\mathbb{P}$ is given by $z_0 = 0$.

On the other side the inclusion $\Omega^1_{X/k}(\log(D)) \to \mathcal{O}_X \oplus \Omega^1_{X/k}(\log(D))$ induces a rational map $h : \mathbb{P}' \dashrightarrow \mathbb{P}$. Let $q : \tilde{Z} \to \mathbb{P}'$ be the blow up along the section given by the projection $\mathcal{O}_X \oplus \Omega^1_{X/k}(\log(D)) \to \mathcal{O}_X$; it resolves the indeterminacy of h and we obtain a morphism $p : \tilde{Z} \to \mathbb{P}$. By construction we obtain $p^*(\mathbb{L}) = q^*(\mathbb{M})(-E)$ where E is the exceptional divisor of $\tilde{Z}$.

The form ω^{sm} induces positive metrics $\|\cdot\|^{sm}_{\mathbb{M}}$ on $\mathbb{M}$ and $\|\cdot\|^{sm}_{\mathbb{L}}$ on $\mathbb{L}$. We denote by $c_1(\mathbb{M})^{sm}$ and $c_1(\mathbb{L})^{sm}$ the corresponding singular first Chern forms. We put on $\mathcal{O}_{\tilde{Z}}(E)$ the metric for which the isomorphism $p^*(\mathbb{L}) \simeq q^*(\mathbb{M})(-E)$ become an isometry. The form $\tilde{\omega}$ induces singular metrics $\|\cdot\|^{s}_{\mathbb{M}}$ on $\mathbb{M}$ and $\|\cdot\|^{s}_{\mathbb{L}}$ on $\mathbb{L}$. We denote by $c_1(\mathbb{M})^s$ and $c_1(\mathbb{L})^s$ the corresponding singular first Chern forms.

The morphism

$$\begin{aligned} \mathcal{O}_Y \oplus f^*(\Omega^1_X(\log(D))) &\longrightarrow \Omega^1_Y(S + f^{-1}(D)) \\ (a, \alpha) &\longrightarrow a\partial(g) + f^*(\alpha) \end{aligned}$$

induces maps $f_1 : Y \to \mathbb{P}'$ and $\tilde{f} : Y \to \tilde{Z}$. By construction $p \circ \tilde{f} = f'$.

Locally on Y, we can write f as $(g_1, \cdots, g_n)$ and $f_1 = (g_1, \cdots, g_n) \times [g' : \frac{g_1'}{g_1} : \cdots : \frac{g_r'}{g_r} : g'_{r+1} : \cdots : g'_n]$.

We apply the first main theorem to the map f_1 the line bundle $\mathbb{M}$ equipped with $\|\cdot\|^s_{\mathbb{M}}$ and the divisor $\mathbb{P}$. Remark that, even if the metric $\|\cdot\|^s_{\mathbb{M}}$ is singular, we can apply the FMT because it is locally integrable.

By the local computation of f_1 we see that $N_{\mathbb{P}}(Y)(r) = \chi(Y)(r) + N^{(1)}_D(Y, r)$. Thus we obtain

$$\int_{-\infty}^{\log(r)} dt \int_{g \le t} f_1^* c_1(\mathbb{M})^s = \chi(Y)(r) + N^{(1)}_D(Y, r) - 2\int_{g=r} \log\|\mathbb{P}\|^s_{\mathbb{M}} d^c g.$$

Consequently we obtain

$$\begin{aligned} &\int_{-\infty}^{\log(r)} dt \int_{g \le t} f'^*(c_1(\mathbb{L})^s) = \\ &= \chi(Y)(r) + N^{(1)}_D(Y, r) - 2\int_{g=r} \log\|\mathbb{P}\|^s_{\mathbb{M}} d^c g - N_E(Y)(r) + 2\int_{g=r} \log\|E\| d^c g + O(1). \end{aligned}$$

We claim that $(\mathbb{L}, Y)(r)$ is smaller then $\int_{-\infty}^{\log(r)} dt \int_{g\leq t} f'^*(c_1(\mathbb{L})^s) + O(\log(H;Y)(r))$. The intersection $(\mathbb{L}, Y)(r)$ can be computed using the metric $\|\cdot\|_{\mathbb{L}}^{sm}$. Outside D, we can find a function h such that $\|\cdot\|_{\mathbb{L}}^{sm}\cdot h = \|\cdot\|_{\mathbb{L}}^{s}$. Thus $c_1(\mathbb{L})^s = c_1(\mathbb{L})^{sm} - dd^c\log(h)$. Computing the two metrics locally, again by compactness of X, we obtain that $h \ll \prod \log^2\|D_i\|$. Consequently

$$\int_{-\infty}^{\log(r)} dt \int_{g\leq t} f'^*(c_1(\mathbb{L})^s) = (\mathbb{L};Y)(r) - \int_{\infty}^{\log(r)} dt \int_{g\leq t} dd^c\log(h) + O(1);$$

by applying Stokes Theorem twice, we find that

$$\int_{\infty}^{\log(r)} dt \int_{g\leq t} dd^c\log(h) = \int_{g=\log r} \log(h) d^c g + O(1).$$

This last term can be bounded as follows

$$\begin{aligned}\int_{g=\log r} \log(h) d^c g &\ll \sum_i \int_{g=\log r} \log(\log^2\|D_i\|) d^c g\\ &\leq \sum_i 2\log\int_{g=\log r} |\log\|D_i\|| d^c g\\ &\leq \sum_i 2\log((\mathcal{O}_X(D_i);Y)(r)) \ll \log((H;Y)(r)).\end{aligned}$$

The claim follows.

We compute now, locally, $\tilde{f}^*(\|E\|)(z)$ and $f_1^*(\|\mathbb{P}\|_{\mathbb{M}}^s)(z)$. Let z be a local coordinate on Y and let ∂_z be the corresponding local generator of the tangent bundle of Y. Define $\tilde{\omega}(z) := f^*(\tilde{\omega})(\partial_z)$. The local expression of f, f_1 etc. implies that $\tilde{f}^*(\|E\|)^2(z) = \frac{\tilde{\omega}(z)}{\tilde{\omega}(z)+|\partial_z g|^2}$ and $\tilde{f}^*(\|\mathbb{P}\|_{\mathbb{M}}^s)^2(z) = \frac{|\partial g|^2}{\tilde{\omega}(z)+|\partial_z g|^2}$.

In order to conclude, we need to find an upper bound for

$$T(r) := \int_{g=r} \log\frac{\tilde{\omega}(z)}{|\partial_z g|^2} d^c g.$$

We can find a function F such that $f^*(\tilde{\omega}) = F dg \wedge d^c g$. The function F will be $|z|^2\times$ smooth in the neighborhood of the poles of g and in general $F(z) = \frac{\tilde{\omega}(z)}{|\partial_z g|^2}$.

Let

$$S(r) := \int_{\infty}^{\log(r)} dt \int_{g\leq t} f^*(\tilde{\omega}).$$

Fubini Theorem gives

$$S(r)' = \int_{-\infty}^{\log(r)} dt \int_{g=t} F d^c g;$$

Thus, cancavity of the log gives

$$\log(S^{(2)}(r)) \geq T(r).$$

The following lemma is well known and elementary (for a proof cf. [GK])

4.4 Lemma. *Let H be a derivable positive increasing function. For every positive ϵ, there exists a subset $E \subset \mathbb{R}$ with $meas(E) \leq \int_{1+\epsilon}^{\infty} \frac{1}{x \log^{1+\epsilon}(x)} dx < \infty$, such that, for every $x \notin E$,*

$$H'(x) \leq H(x) \log^{1+\epsilon}(H(x)).$$

We apply Lemma 4.4 twice and we find that

$$T(r) \leq \log(S(r) \log^{1+\epsilon}(S(r)) \log^{1+\epsilon}(S(r) \log^{1+\epsilon}(S(r)))).$$

We will conclude if we find an upper bound for $S(r)$.

The following equality holds

$$-dd^c \log(\log^2(\|D_i\|)) = \frac{d\|D_i\| \wedge d^c\|D_i\|}{\|D_i\|^2(\log \|D_i\|)^2} + \frac{1}{|\log(\|D_i\|)|} \cdot c_1(\mathcal{O}(D_i));$$

by the compactness of X, the last term on the right hand side is uniformly bounded; thus we can find a constant A such that

$$\tilde{\omega} \leq A\omega - \sum_i dd^c \log(\log^2(\|D_i\|)).$$

Thus, again by applying Stokes,

$$S(r) \ll (H;Y)(r) - 2\int_{g=r} \log(\log^2(\|D_i\|))d^c g + O(1).$$

Since we can suppose that $\|D_i\| < \epsilon$ we conclude.

5 Currents associated to families of curves.

Let X be a projective variety (reduced and irreducible) and H an ample line bundle equipped with a smooth positive metric. We will now show how to associate a closed positive current to the situation we are interested in.

In the analytic situation we start with a map from a parabolic Riemann surface to X and the diophantine statement we are interested in upper bounds *uniform in* r. Roughly speaking we have maps from the Riemann surfaces $\{z \in Y \ / \ g(z) < \log(r)\}$ and we look for uniform upper bounds for their areas (or better: the average of the areas over them) with respect to some hermitian line bundle, in terms of their Euler characteristic and the (set theoretical) intersection with the divisor at infinity. Thus we can take a sequence of r's which goes to infinity and do not satisfy the wanted inequality and eventually find a contradiction.

Similarly, in the algebraic case, we want to give uniform upper bounds of the intersection (the height!) of the closed curves in X in terms of the Euler characteristic of their

normalization and the (set theoretic) intersection with the divisor at infinity. Again we take a sequence of smooth projective curves which do not satisfy the inequality and find a contradiction.

We will show now, that in both situations we can associate to the involved sequences a closed positive current T on X (and on other varieties constructed during the proof). The proof of the theorem will work with the properties of T and only at the end, the definition of it will give the statement in the analytic or in the geometric case.

We will consider two situations:

a) *The analytic situation* S^{an}: A parabolic Riemann surface and a holomorphic map $f\colon Y \to X$.

b) *The algebraic situation* S^{alg}: A sequence of smooth projective curves $\{Y_n\}_{n\in\mathbb{N}}$ and algebraic maps $f_n : Y_n \to X$.

Before we start the construction, we have to show that the currents associated to S^{an}, even if they are not closed, they are "closed enough".

Let (Y, g) be a parabolic Riemann surface equipped with a positive singularity. Let $f : Y \to X$ be a holomorphic map.

We show now that the intersection of Y with exact forms is essentially irrelevant.

5.1 Lemma. *Let* $1/2 > \epsilon > 0$. *Let* α *be a smooth exact* $(1,1)$ *form on* X *then*

$$(\alpha, Y)(r) = O_\epsilon(((H;Y)(r))^{1/2}\log^{1-\epsilon}(r(H;Y)(r))) \;\; //$$

where the involved constants depend only on ϵ.

Proof: Since $(\cdot, Y)(r)$ is a positive current, it will suffice to prove the theorem when α is $\overline{\partial}\beta$ for a smooth $(1,0)$ form β.

By Stokes theorem

$$\begin{aligned}(\alpha;Y)(r) &= \int_0^r \frac{dt}{t}\int_{g=t}\beta \\ &= \int_{-\infty<g\leq\log(r)} dg\wedge\beta.\end{aligned}$$

Since X is compact, we have that $\beta\wedge\overline{\beta} \ll c_1(H)$. Consequently, Cauchy–Schwartz inequality gives

$$\left|\int_{-\infty<g\leq\log(r)} dg\wedge\beta\right| \leq 2\pi\left|\int_{-\infty<g\leq\log(r)}\beta\wedge\overline{\beta}\right|^{1/2}\cdot\left|\int_{-\infty<g\leq\log(r)} dg\wedge d^c g\right|^{1/2}.$$

We apply again 4.4 and we obtain that, outside a set of finite Lebesgue misure,

$$\left|\int_{-\infty<g\leq\log(r)}\beta\wedge\overline{\beta}\right| \ll (H,Y)(r)\log^{1+\epsilon}((H;Y)(r)).$$

Since (again by Stokes) $\int_{-\infty<g\leq\log(r)} dg\wedge d^c g \leq C\log(r)$, for a suitable C, we conclude.

5.2 Remark. In the algebraic setting, if Y is a smooth algebraic curve and $f\colon Y \to X$ is a map, the current $\alpha \to \int_Y f^*(\alpha)$ is closed by Stokes theorem. We observe that the lemma above tells us that, up a negligible term, the current $(\cdot; Y)(r)$ is closed. Consequently, up to this negligible term, this is another analogy between the two situations, we will see now how to push forward this.

Let (X, H) as above. In the analytic situation consider the set of currents

$$\begin{aligned} T_r : A^{1,1}(X) &\longrightarrow \mathbb{R} \\ \alpha &\longrightarrow \frac{1}{(H;Y)(r)} \int_0^r \frac{dt}{t} \int_{g<t} f^*(\alpha); \end{aligned}$$

in the geometric situation consider the set of currents

$$\begin{aligned} T_n : A^{1,1}(X) &\longrightarrow \mathbb{R} \\ \alpha &\longrightarrow \frac{\int_{Y_n} f_n^*(\alpha)}{(H;Y_n)}. \end{aligned}$$

In both situations they are families of positive currents bounded for the standard norm on $A^{1,1}$ (but also for the L^∞ norm); consequently we can extract from them a sequence converging, in the weak topology, to a positive current T.

In the algebraic situation the current T is closed because it is limit of closed currents.

In the analytic case, due to Lemma 5.1, we can choose the sequence in order to obtain a sequence T_{r_n} such that $dT_{r_n} \to 0$. Observe that, even if the involved map f is algebraic, the height of r at least $A\log(r)$ (for a suitable A) consequently the Lemma apply.

5.3 Definition. *The closed positive current T constructed above, will be called* the current *associated to the (geometric or analytic) situation.*

Observe that T is non zero because $T(c_1(H)) = 1$.

5.4 Remark. the article *the* in the definition is not completely correct. Indeed the current T depends on the choice of the subsequences involved. The reader will check that we will use only properties which hold for *every* sequence as above.

5.5 Remark. (Important) Since the current T is closed, we can unambiguously compute it on a Cartier divisor R of X: put an arbitrary smooth metric on $\mathcal{O}_X(R)$ and define $T(R) := T(c_1(\mathcal{O}_X(R))$; this number do not depend on the chosen metric. Of course, since T is positive, if R is ample (resp. nef) then $T(R) > 0$ (resp. $T(R) \geq 0$). One should see T as a class in the dual of the positive cone of $NS(X)_{\mathbb{R}}$ and interpret $T(R)$ as an intersection number.

6 First approach to the theorems.

In this section we will explain the proof by McQuillan of theorems 3.1 and 3.2. We recall that we are assuming the reductions made after the statements. We begin by fixing some notations: $\pi : \mathbb{P} := \mathrm{Proj}(\Omega^1_X(\log(D))) \to X$ will be the projective bundle associated to the sheaf of differentials with logarithmic poles around D. Let $\mathbb{L}$ be the tautological bundle over $\mathbb{P}$. We denote by F a smooth fibre of p. We fix an ample divisor H on X; observe that we may suppose that there exists ϵ' such that, for every curve Y not contained in a fibre, $(Y, K_X(D)) \geq \epsilon'(Y; H)$ (and similarly in the analytic setting).

The theorems are proved by contradiction.

a) *Algebraic situation*: We suppose that there is a sequence of smooth projective curves and morphisms $f_n : Y_n \to X$ such that

$$\lim_{n\to\infty} \frac{N_D^{(1)}(Y_n) + \chi(Y_n)}{(H; Y_n)} < \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{[Y_n : B]}{(H; Y_n)} = 0. \tag{6.1.1}$$

Consequently we can construct a closed and positive current T on X associated to the sequence. We can rise each map f_n to a map $f'_n : Y_n \to \mathbb{P}$. Each f'_n give rise to a closed positive current

$$\begin{aligned} T'_n : & A^{(1,1)}(\mathbb{P}) \longrightarrow \mathbb{R} \\ & \alpha \longrightarrow \frac{1}{(H; Y_n)} \int_{Y_n} f'^*_n(\alpha). \end{aligned}$$

Because of the hypothesis 6.1.1, we can extract from the sequence above a subsequence converging to a closed positive current T' on $\mathbb{P}$. By construction we have that $\pi_*(T') = T$.

The theorem will be proved if we show that

$$T'(\mathbb{L} - \pi^*(K_{X/D}(D))) \geq 0.$$

b) *Analytic situation*: We suppose that there exists a sequence of real numbers r_n such that

$$\lim_{n\to\infty} \frac{N_D^{(1)}(Y)(r_n) + \chi(Y)(r_n)}{(H, Y)(r_n)} < \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{[Y; B](r_n)}{(H; Y)(r_n)} = 0. \tag{6.2.1}$$

Thus, again we can associate to this sequence the closed positive current T on X. We can rise the map f to a map $f' : Y \to \mathbb{P}$. Each r_n give rise to a closed positive current

$$\begin{aligned} T'_n : & A^{(1,1)}(\mathbb{P}) \longrightarrow \mathbb{R} \\ & \alpha \longrightarrow \frac{1}{(H; Y)(r_n)} \int_0^{r_n} \frac{dt}{t} \int_{g\leq t} f'^*(\alpha). \end{aligned}$$

Because of the hypothesis 6.2.1, we can extract from the sequence above a subsequence converging to a closed positive current T' on $\mathbb{P}$. By Lemma 5.1, if we choose suitably the subsequence, the currents will converge to a closed current because we can suppose that $\mathbb{L} + \pi^*(H)$ is ample. By construction we have that $\pi_*(T') = T$.

Again, the theorem will be proved if we show that

$$T'(\mathbb{L} - \pi^*(K_{X/D}(D))) \geq 0.$$

6.3 Remark. Observe that 6.1.1 and 6.2.1 imply that if V is a vertical divisor on X (the map $p|_V : V \to B$ is not dominant) then $T(V) = 0$.

Let $p : X \to B$ the given fibration; $S := \{P_i\}$ be the set of singular points of the fibers of p. By the reduction assumed, we may suppose that each P_i belongs to exactly two smooth component of a fibre which intersect properly on P_i; denote these two components B^i and C^i. We will denote by I_S the ideal sheaf of the subscheme defined by S on X. The fibration p give rise to an exact sequence

$$0 \to p^*(\Omega^1_B) \longrightarrow \Omega^1_X(\log(D)) \longrightarrow I_S \otimes K_{X/B}(D) \to 0. \tag{6.4.1}$$

Let $b : \tilde{X} \to X$ be the surface obtained taking the blow up of X over the P_i's and $E := \sum_i E_i$ be the exceptional divisor (E_i being the exceptional divisor over P_i). The exact sequence 6.4.1 give rise to an injection $\iota : \tilde{X} \to \mathbb{P}$; by construction

$$\iota^*(\mathbb{L}) = b^*(K_{X/B}(D))(-E).$$

Denote by Δ the image of ι; it is a divisor on $\mathbb{P}$. Observe that the fibre of π over the P_i's is contained in Δ. Denote by U the open set $\mathbb{P} \setminus \Delta$.

One of the main tools of the proof is the

6.5 Proposition. *We have that*

$$\pi_*(\mathbb{I}_U T') = 0.$$

The proposition above is not a surprise! Essentially it tells us the following: The area of a curve on X has a vertical and an horizontal component (with respect to p); If the theorems are false, then we can find a sequence of curves for with the vertical component of the area is much bigger then the horizontal one. Thus the limit of the curves must be vertical.

Proof: We fix Khäler forms ω on X and η on B. In order to prove the proposition it suffices to prove the following: Let V be an open set of $\mathbb{P}$ such that $\overline{V} \cap \Delta = \emptyset$ ($\overline{V}$ being the closure of V in the Euclidean topology), then $\pi_*(\mathbb{I}_V T) = 0$. To prove this we will show the following: there exists a constant A_V (depending on V and the metrics) such that the following holds: if W is an open Riemann surface and $h : W \to X$ is an holomorphic map such that the image of $h' : W \to \mathbb{P}$ is contained in V, then

$h^*(\omega) \leq A_V(p\circ h)^*(\eta)$. The conclusion follows because T (resp. T') is zero on the fibres of p (resp. on the fibres of $p \circ \pi$).

Fix such a V, Observe that $\pi(\overline{V})$ is a closed set of X which do not contains the singular points of the fibers P_i. By compactness of X, we can cover $\pi(\overline{V})$ by a finite set of disks B_j not containing the P_i's. We may restrict our attention to each of the B_j: thus we may suppose that:
– $X = \{(z, w) \in \mathbb{C}^2 \ / \ |z| < 1 \ |w| < 1\}$, $B = \{z \in C \ /|z| < 1\}$ and $p(z, w) = z$;
– $\omega = \sqrt{-1}(dz \wedge d\overline{z} + dw \wedge d\overline{w})$ and $\eta = \sqrt{-1}(dz \wedge d\overline{z})$;
– $D = \{w = 0\}$ and the exact sequence 6.4.1 is the split exact sequence

$$0 \to \mathcal{O}_X dz \longrightarrow \mathcal{O}_X dz \oplus \mathcal{O}_X \frac{dw}{w} \longrightarrow \mathcal{O}_X \frac{dw}{w} \to 0;$$

– consequently $\mathbb{P} = X \times \mathbb{P}^1$ and $\Delta = X \times \{[0 : 1]\}$; we may then suppose that there exists a positive constant a such that $V \subseteq \{(z, w) \times [x : y] \ / \ |x|^2 > a|y|^2\}$.
– $W := \{z \ / \ |z| < 1\}$ and $h(z) = (h_1; h_2)$ and $h'(z) = (h_1; h_2) \times [h'_1 : \frac{h'_2}{h_2}]$.

The image of W via h' is contained in V, we have that $|h'_1(z)|^2 > a \left|\frac{h'_2}{h_2}\right|^2$. Thus $\frac{|h'_2(z)|^2}{|h'_1(z)|^2} < \frac{1}{a}$. Since $h^*(\omega) = \sqrt{-1}(|h'_1|^2 + |h'_2|^2)dz \wedge d\overline{z}$ and $h^*(\eta) = \sqrt{-1}(|h'_1|^2)dz \wedge d\overline{z}$ the proposition follows.

Since the line bundle $\mathbb{L}$ is nef, as far as we are interested, we may work as if the current T is supported on Δ. Indeed, by the proposition, $\mathbb{I}_U T = T - \mathbb{I}_\Delta T$ is a current which intersect positively $\mathbb{L}$ and whose intersection with $\pi^*(K_X(D))$ is zero. Moreover, again, as far as we are interested, via the proposition below, we can even suppose that it is the push forward of a current on Δ.

6.6 Proposition. *Let X be a smooth variety and E be a smooth divisor on it. Let T be a closed positive current of type $(1, 1)$ over X. Let $\iota\colon E \to X$ be the inclusion. Then there is a current S on E such that*

$$\mathbb{I}_E \cdot T = \iota_*(S).$$

It is evident that S is positive and closed.
Proof: In order to prove the proposition, we need to show that, if ω is a form such that $\iota^*(\omega) = 0$ then $\mathbb{I}_E \cdot T(\omega) = 0$.

Locally on X we can suppose that $z_1, \ldots, z_n$ are coordinates on X and $E = \{z_n = 0\}$. The theorem is proved if we show that for every i and for every C^∞ function a with compact support, we have that $\mathbb{I}_E \cdot T(a \cdot dz_n \wedge d\overline{z}_i) = \mathbb{I}_E \cdot T(a \cdot dz_i \wedge d\overline{z}_n) = 0$.

Let ψ be a positive function with compact support which is 1 on the support of a. Since $\mathbb{I}_E \cdot T$ is positive, the Cauchy–Schwartz inequality gives

$$|\mathbb{I}_E \cdot T(a \cdot dz_n \wedge d\overline{z}_i)|^2 \leq |\mathbb{I}_E \cdot T(\psi \cdot \sqrt{-1}dz_n \wedge d\overline{z}_n)|^2 \cdot |\mathbb{I}_E \cdot T(a \cdot \sqrt{-1}dz_i \wedge d\overline{z}_i)|^2;$$

consequently it suffices to show that $|\mathbb{I}_E \cdot T(\psi \cdot \sqrt{-1} dz_n \wedge d\overline{z}_n)|^2 = 0$. Since T is of type $(1,1)$ and closed, $\mathbb{I}_E \cdot T(\psi dd^c(|z_n|^2) = \mathbb{I}_E \cdot T(|z|^2 dd^c\psi) = 0$. But since $dd^c|z_n|^2 = \frac{\sqrt{-1}}{2\pi} dz_n \wedge d\overline{z}_n$; the conclusion follows.

We apply the proposition above with $X = \mathbb{P}$ and $E = \Delta$. Thus, there is a closed positive current S on Δ such that $\mathbb{I}_\Delta T' = \iota_*(S)$. Observe that, by functoriality, $b_*(S) = T$.

The proof of the theorem will be achieved if we can prove that $S(-E_i) \geq 0$. In particular, if X is smooth over B and the divisor D is étale over B then the theorem is proved; for instance the isotrivial case (Nevanlinna Second Main Theorem) is proved.

We reduced the difficulty to a local problem around the singular points of the fibres. Most of the remaining of the proof will be of local nature, but will will notice that one main point will be of global nature.

The proof proceed by working on coverings of X; Let $Q_1, \ldots, Q_r$ be the points of B where p is not smooth. we fix another point Q on B. For every m sufficiently big, we can find a covering $B_m \to B$ which is cyclic of order m, totally ramified over $Q_1, \ldots, Q_r$ and étale over $B \setminus \{Q, Q_1, \ldots, Q_r\}$.

In the algebraic situation, the normalization $Y_{n,m}$ of the curves $Y_n \times_B B_m$ are such that $\frac{\chi(Y_{n,m})}{m} \leq \chi(Y_n) + A$ (with A independent on m).

In the analytic situation, the normalization Y_m of $Y \times_B B_m$ is a parabolic Riemann surface, with as positive singularity, the pull back of g (which we will denote by g_m). Also in this situation $\frac{\chi(Y_m)(r)}{m} \leq +\chi(Y)(r) + A$.

Consider the surface $X_m := X \times_B B_m \stackrel{g_m}{\to} X$. Let $p_m : X_m \to B_m$ be the second projection.

The surface X_m is normal and $\mathbb{Q}$–factorial. Denote by D_m the divisor $g_m^*(D)$. For every i there is a unique singular point R_i over the P_i. For every i, there is an analytic neighborhood of R_i isomorphic to an analytic neighborhood of the surface $Z^m = XY$ (R_i corresponds to $(0,0,0)$). Denote by U_m the open surface $X_m \setminus \{R_i\}$.

In the next subsection we will justify the following properties:
– There exists a normal $\mathbb{Q}$ factorial variety $\mathbb{P}_m$ with a $\mathbb{Q}$ line bundle $\mathbb{L}_m$ and a projection $\pi_m : \mathbb{P}_m \to X_m$. Over U_m there is a canonical isomorphism $i_m : \mathbb{P}_m|_{U_m} \leftrightarrow \mathbb{P}(\Omega_{U_m}(\log(D_m))$ whose pull back of the tautological line bundle is the restriction of $\mathbb{L}_m$.
– The analogue of the exact sequence 6.4.1 over U_m give rise to an inclusion $U_m \to \mathbb{P}(\Omega_{U_m}(\log(D_m))$; Let $\tilde{X}_m$ be the closure of the image. Let $h_m : \tilde{X}_m \to X_m$ be the projection and $\iota_m : \tilde{X}_m \to \mathbb{P}_m$ be the inclusion. The surface $\tilde{X}_m$ is again normal and $\mathbb{Q}$–factorial. The preimage of R_i is a Weil divisor and coincide with the fibre over R_i of π_m. Denote it by E_i^m. Moreover $\iota_m^*(\mathbb{L}_m) = g_m^*(K_{X_m/B_m}(D_m))(-\sum_i E_i^m)$ (this is an equality between $\mathbb{Q}$–divisors). Denote by V_m the open set $\mathbb{P}_m \setminus \iota(\tilde{X}_m)$ (remark that V_m is smooth).
– For every R_i let B_m^i and C_m^i be the two components of the fibre of p_m meeting on R_m. We have that $h_m^*(B_m^i) = \tilde{B}_m^i + E_i^m$ and $h_m^*(C_m^i) = \tilde{C}_m^i + E_m^i$ where $\tilde{B}_m^i$ is birational to

B_m^i and $\tilde{C}_i^m$ to C_m^i respectively.
– Let $f_{n,m} : Y_{n,m} \to X_m$ (resp. $f_m : Y_m \to X_m$ in the analytic situation); taking, if necessary, a subsequence of the curves $Y_{n,m}$ (resp. of the r_n), we can construct, as before, a closed positive current T_m on X_m (resp. T'_m on $\mathbb{P}_m$) such that $g_{m,*}(T_m) = T$; observe that we have to normalize dividing by m. The value of the currents T_m and T'_m on the fibres of p_m and of $\pi_m \circ p_m$ respectively, is zero.
– We can find a constant A_m (depending on m) such that, in the algebraic situation

$$\frac{(\mathbb{L}_n, Y_{n,m})}{m} \le N_D^1(Y_n) + \chi(Y_n) + A_m[Y_n; B];$$

and in the analytic situation

$$\frac{(\mathbb{L}_n, Y_m)(r)}{m} \le N_D^1(Y_n)(r) + \chi(Y_n)(r) + A_m[Y; B](r);$$

thus the theorem will be proved if we show that there exists a constant A (independent on m) such that

$$T'_m(\mathbb{L}_m - \pi_m^*(K_{X_m/B_m}(D_m))) \ge \frac{A}{m}.$$

Since the singular points of $\mathbb{P}_m$ are contained in the image of ι_m; we can prove, exactly as before, that

$$\pi_{m,*}(\mathbb{I}_{V_m} T') = 0.$$

Since $\mathbb{P}_m$ is $\mathbb{Q}$–factorial and $\iota_m(\tilde{X}_m)$ is a divisor, the analogue of 6.6 holds; thus there is a current S_m on $\tilde{X}_m$ such that $\mathbb{I}_{\iota_m(\tilde{X}_m)} T'_m = \iota_*(S_m)$. Moreover $h_{m,*}(S_m) = T_m$.

The theorem is proved if we show that there is a constant A such that

$$S_m(-E_i^m) \ge \frac{A}{m}.$$

Computing on the smooth part of X_m, we find that $g_m^*(B^i) = mB_m^i$ and $g_m^*(C^i) = mC_m^i$.

On $\tilde{X}_m$, since $g_m^*(B^i) = mB_m^i$ and $(g_m \circ h_m)_*(S_m) = T$ we have that $S_m(h_m^*(B_m^i)) = 0$ (cf. remark 6.3). Thus $S_m(-E_m^i) = S_m(\tilde{B}_m^i)$.

6.7 Lemma. *Let B be an effective divisor on a projective variety X and R be a closed positive current on X such that $\mathbb{I}_B R = 0$, then $R(B) \ge 0$*

We will postpone the proof of the lemma in the next subsection.

Because of the lemma, applied to $\mathbb{I}_{\tilde{X}_m \setminus \tilde{B}_m^i} S_m$, we have that $S_m(\tilde{B}_m^i) \ge \mathbb{I}_{\tilde{B}_m^i} S_m(\tilde{B}_m^i)$. Since $\tilde{B}_m^i$ and $\tilde{C}_m^i$ are disjoint, $\mathbb{I}_{\tilde{B}_m^i} S_m(\tilde{B}_m^i) = \mathbb{I}_{\tilde{B}_m^i} S_m(\tilde{B}_m^i - \tilde{C}_m^i)$.

The divisor $\tilde{B}_m^i - \tilde{C}_m^i$ is $(h_m \circ g_m)^*(\frac{B^i - C^i}{m})$, thus

$$S_m(-E_i^m) \ge \frac{(h_m \circ g_m)_*(\mathbb{I}_{\tilde{B}_m^i} S_m)(B_i - C_i)}{m}.$$

Since $(h_m \circ g_m)_*(S_m) = T$, the following easy remark, applied to the couples $(T; (h_m \circ g_m)_*(\mathbb{I}_{\tilde{B}^i_m} S_m)$ and $(T; (h_m \circ g_m)_*(\mathbb{I}_{\tilde{X}_m \setminus \tilde{B}^i_m} S_m))$, allows to conclude:

Let C be a divisor on a variety X and T a closed positive current on X; then there exists a constant A depending only on C and T for which the following holds: for every closed positive current T_1 with $T \geq T_1$, we have that $T(C) \geq T_1(C) + A$ (proof: take an ample bundle H such that $C + H$ is ample and compute T and T_1 on $C + H$).

6.8 *Extension of some results to singular varieties.* In this subsection, we will explain how to extend the results we need to the singular varieties involved in the proof. As explained in the introduction, a natural approach to the proof is via integration on stacks. Unfortunately, even for this easy orbifold case, we need to develop the entire theory of integration on stacks; this is why we prefer to deal with singular varieties.

A systematic approach to the tautological inequality and the other needed results may be quite complicate, in particular it is not easy to find the minimal hypotheses. This is why we develop just what we need. Moreover this subsection will be sketchy.

Metrized line bundles on singular varieties: Let X be a reduced irreducible projective variety. Let $\mathcal{L}$ be a line bundle on it. A continuous metric on $\mathcal{L}$ is a metric on the fibres of it which varies continuously for the Euclidean topology. We will say that a metric is *smooth* if locally we can embed X in a smooth variety W, $\mathcal{L}$ is the restriction of a line bundle $\mathcal{L}_{\mathcal{W}}$ on W and the metric is the restriction of a smooth metric on $\mathcal{L}_{\mathcal{W}}$. We see that this is equivalent to ask that, for every smooth variety Y and map $f : Y \to X$, the induced metric on the line bundle $f^*(\mathcal{L})$ is smooth. A (local) section of $\mathcal{L}$ is said to be smooth if, locally it is the restriction of a section on a smooth variety.

Observe that the sheaf $\Omega^{1,1}_X$ has a meaning on X: Ω^1_X exists, and $\overline{\Omega}^1_X$ is its conjugate; thus $\Omega^{1,1} := \Omega^1_X \otimes \overline{\Omega}^1_X$. A $(1,1)$ form is said to be *smooth* if, locally it is the restriction of a smooth form of a smooth variety. Similarly for functions.

Every line bundle on X is difference of very ample line bundles, thus every line bundle on X admits a smooth metric.

Given a line bundle $\mathcal{L}$ on X equipped with a smooth metric, we can define its first Chern form in the following way: take a (local) smooth section f and $c_1(L) := -dd^c \log \|f\|^2$ outside the zeroes of f. Observe that dd^c is well defined on smooth functions and that $c_1(L)$ is a smooth $(1,1)$ form on X. If we change the metric on $\mathcal{L}$ by another smooth metric, the first Chern form varies by the dd^c of a smooth function on X.

We gave examples to show that we can define all the objects we need as restriction of similar objects defined over smooth varieties: in particular we can define also the currents on X and we can give a meaning to closed and positive currents.

Construction of $\mathbb{P}_m$ and related objects: The surface X_m is smooth except on the points R_i. Near the R_i it is isomorphic to the surface $Z^m = XY$. Let $D_{\zeta,\xi} := \{(\zeta, \xi) \ /|\zeta| < 1;\ |\xi| < 1\}$. Let μ_n the cyclic group of the m–roots of the unity and let θ_m be a generator of it; it acts on $D_{\zeta,\xi}$ with the action $\zeta \to \theta_m \zeta$ and $\xi \to \theta_m^{-1} \xi$. For every i, there is a neighborhood V_i of the singular point R_i on X_m, isomorphic to

$D_{\zeta,\xi}/\mu_m$. Observe that we may suppose that V_i do not intersect the divisor D_m. The cyclic group μ_m acts on the cotangent sheaf of $D_{\zeta,\xi}$ thus on $\mathbb{P}(\Omega^1_{D_{\zeta,\xi}})$. Denote by $\mathbb{P}_{D_m}$ the quotient $\mathbb{P}(\Omega^1_{D_{\zeta,\xi}})/\mu_m$. There is a natural projection $\mathbb{P}_{D_m} \to V_i$ and the restriction of $\mathbb{P}_{D_m}$ to $V_i \setminus \{R_i\}$ is isomorphic to the restriction of $\mathrm{Proj}(\Omega^1_{X_m}(\log(D_m))$. Thus we can glue together the restriction of $\mathrm{Proj}(\Omega^1_{X_m}(\log(D_m))$ to $X_m \setminus \{R_i\}$ and $\mathbb{P}_{D_m} \to V_i$ to obtain a variety $\pi_m : \mathbb{P}_m \to X_m$ which is normal and $\mathbb{Q}$–factorial by construction (locally it is quotient of a smooth variety by a finite group). One easily verify that $\mathbb{P}_m$ is projective and equipped with a $\mathbb{Q}$–line bundle $\mathbb{L}_m$ which has the searched properties. Observe that $\mathbb{P}_m$ has only isolated singular points.

The extension of the tautological inequality to singular variety is straightforward: Let X_m^{sm} be a desingularization of X_m, $\mathbb{P}_m^{sm} \to X_m^{sm}$ be the corresponding projective bundle of the logarithmic differentials and $\mathbb{L}_m^{sm}$ the tautological bundle over it. Since $\mathbb{P}_m$ and $\mathbb{P}_m^{sm}$ are birational, there exists a smooth variety Z_m, a commutative diagram

$$\begin{array}{ccc} Z_m & \xrightarrow{a} & \mathbb{P}_m^{sm} \\ \big\downarrow b & & \big\downarrow \\ \mathbb{P}_m & \xrightarrow{p_m} & B_m \end{array}$$

where the morphisms a and b are birational and a divisor A on Z_m such that $b^*(\mathbb{L}_m) = a^*(\mathbb{L}_M^{sm}) + A$. Since the divisor A is vertical, (contained over a fibre of $p_m \circ b$) and the tautological inequality holds on X_m^{sm}, the needed tautological inequality holds on X_m.

The construction of the currents T_m and T'_m is similar to the construction of the currents T and T': everything is defined to let the construction work. Observe that the intersection of both T_m and T'_m with a vertical divisor is zero.

To prove 6.7 we need the analogue of Stokes theorem for currents:

6.8 Proposition. *Let T be a closed positive current on a projective variety X. Let f be a smooth function on it. Then for almost all ϵ we there exists a closed positive current T_ϵ on $X_\epsilon := \{z \in X \ / \ f(z) = \epsilon\}$ such that the following equality holds for every smooth form ω:*

$$\int_{\{f\leq\epsilon\}} T \wedge d(\omega) = \int_{X_\epsilon} T_\epsilon \wedge \omega.$$

Sketch of Proof: We can find a sequence of smooth closed currents T_n such that $T_n \to T$ in the weak topology. By Fubini theorem, we have that, for suitable a and b in $\mathbb{R}$

$$T(df \wedge d^c f) = \lim_{n\to\infty} T_n(df \wedge d^c f) = \lim_{n\to\infty} \int_a^b dt \int_{X_t} T_n \wedge d^c f;$$

thus, for almost all $\epsilon \in [a;b]$ the integrals $\int_{X_t} T_n \wedge d^c f$ are uniformly bounded. Consequently, for almost all $\epsilon \in [a;b]$ the measures $(T_n \wedge d^c f)|_{X_\epsilon}$ on X_ϵ converge to a measure $T_\epsilon \wedge d^c f$. If ϵ is outside the "bad set",the classical Stokes theorem applied to the smooth

closed currents gives

$$\int_{f\le\epsilon} T\wedge d(\omega)=\lim_{n\to\infty}\int_{f\le\epsilon} T_n\wedge d(\omega)=\lim_{n\to\infty}\int_{X_\epsilon} T_n\wedge\omega=\int_{X_\epsilon} T_\epsilon\wedge\omega.$$

Now we can give the
Sketch of Proof of 6.7: Fix a smooth metric on $\mathcal{O}_X(B)$. Since $\mathbb{I}_B R=0$, by definition

$$R(B)=\lim_{\epsilon\to 0}\int_{\|B\|\ge\epsilon} R\wedge c_1(\mathcal{O}_X(D).$$

By Stokes theorem 6.8, for almost all ϵ

$$\int_{\|B\|\ge\epsilon} R\wedge c_1(\mathcal{O}_X(D))=-\int_{\|B\|\ge\epsilon} R\wedge dd^c\log\|B\|^2=\int_{\|B\|=\epsilon} R_\epsilon\wedge\frac{d^c\|B\|^2}{\epsilon^2}\ge 0.$$

The conclusion follows.

7 Second approach to the theorems.

In this section we will sketch the approach by Yamanoi to the main theorems 3.1 and 3.2.

The Yamanoi approach is via the Ahlfors theory and works directly on the moduli space of pointed stable curves of genus zero. The complete proof requires a big machinery and is quite involved thus we refer to the original paper [YA3] for the general statements. We will give here a simplified proof, in the spirit of Yamanoi paper, in the first non trivial case. The main ideas and difficulties appear already here and we think that this case, and its proof, may help to understand the general case.

The first step is the reduction to the case when X is a blow up of $\mathbb{P}^1\times B$. This reduction goes back to Elkies [EL].

7.1 Proposition. *Suppose that 3.1 and 3.2 hold when X is a blow up of $\mathbb{P}^1\times B$. Then 3.1 and 3.2 hold in general.*

Sketch of Proof: Let $(X;D)$ as in theorems 3.1 and 3.2. changing X by a birational model of it, if necessary, we may suppose that there is a generically finite morphism $g:X\to Z:=\mathbb{P}^1\times B$ (commuting with p) and a simple normal crossing divisor H on $\mathbb{P}^1\times B$ such that $g^*(K_Z(H))=K_X(D)+G$; where G is a suitable effective divisor on X and (set theoretically) $g^{-1}(H)=D+G$.

Suppose that $f:Y\to X$ is a morphism from a curve, then since 3.1 or 3.2 holds for (Z,H), (we omit r in the analytic case) the inequality $(K_Z(H);Y)\le N_H^{(1)}(Y)+\chi(Y)+\epsilon(K_Z(H);Y)+\ldots$ holds. Thus

$$(K_X(D);Y)+(G;Y)\le N_D^{(1)}(Y)+N_G^{(1)}(Y)+\chi(Y)+\epsilon(K_X(D+G);Y))+\ldots.$$

We conclude because $N_G^{(1)}(Y) \leq (G;Y)$ and $K_X(D)$ is big.

8 Ahlfors approach to SMT.

8.1 *Quick review of Ahlfors theory.* Suppose that F and G are two bordered Riemann surfaces having finite Euler Poincaré characteristic (eventually the boundary may be empty). Suppose that $f : F \to G$ is an analytic finite morphism such that $f(\partial F) \subset \partial G$ then the classical Hurwitz formula holds:

$$\chi(F) = \deg(f)\chi(G) + \sum_{P \in Ram(f)} (Ram_P(f) - 1)$$

where $Ram(f)$ is the set of ramification points of f and $Ram_P(f)$ is the ramification index of f at P. Observe that we are using the convention that $\chi(point) = -1$ or that $\chi(\mathbb{P}^1) = -2$.

The first part of the Ahlfors theory is a generalization of this formula when one removes the condition on the boundaries. Let G^o be the interior of G; the set of points of the boundary of F whose image is contained in G^o is called the *relative boundary of* f. Suppose that H is a Riemann surface and η a pseudometric on it (i.e a smooth $(1,1)$ form which is positive everywhere but a finite set of points where it vanishes); If U is a domain in H we denote by $A(U, \eta)$ the area of U with respect to η; if β is a Jordan curve on H we denote by $\ell(\beta, \eta)$ the length of β with respect to the measure defined by η; observe that they are both positive numbers.

We introduce a smooth positive metric ω on G in such a way that $A(G; \omega) < \infty$. The *mean sheet number of* f will be the number $S_f := \frac{A(F; f^*(\omega))}{A(G;\omega)}$. Observe that if f is non ramified and unbordered, then S_f is the degree of f. If U is a domain in G then we define the *sheet number of* U *with respect to* f by $S_f(U) := \frac{A(f^{-1}(U); f^*(\omega))}{A(U;\omega)}$. Similarly, if β is a Jordan curve on G, then we define the *the sheet number of* β by $L_f(\beta) := \frac{\ell(f^{-1}(\beta); f^*(\omega))}{\ell(\beta,\omega)}$. We denote by L_f the length of the relative boundary of f with respect to $f^*(\omega)$. A morphism $f : F \to G$ will be said to be *quasifinite* if it has finite fibres. The first main theorem of Ahlfors theory is

8.1 Theorem. *Let G be a bordered Riemann surface, equipped with a positive metric ω. Let U be a domain and β be a Jordan curve on G. Then there exist positive constants h and k depending only on the metric and on U and β respectively for which the following holds: For every quasifinite morphism $f : F \to G$ from a bordered Riemann surface to G we have the following inequalities*

$$|S_f - S_f(U)| \leq hL_f \quad \text{and} \quad |S_f - L_f(\beta)| \leq kL_f.$$

For a proof we refer to [AH], to [HA] or to [NE]. What is very important in the theorem above is that the constants h and k depend only on U and β (and on the

metric ω) *but not on* F and f. The second main theorem of Ahlfors theory is the following

8.2 Theorem. *Suppose that* G *and* U *is as in the previous theorem, then there is a constant* $h > 0$ *depending only on* U *(and the metric) such that, for every finite covering* $f : F \to G$ *we have that*

$$\max(\chi(f^{-1}(U)); 0) \geq \chi(U)S_f - hL_f.$$

In the sequel we will denote by a^+ the number $\max(a, 0)$.

8.4 *Ahlfors proof of SMT.* We will briefly show how to deduce a form of the SMT from Ahlfors theorems. We will be a little bit sketchy because these things are classical and well kown by experts; we recall them here for reader's convenience and to point out the analogies and the differences within the isotrivial and the non isotrivial cases. Here and in the following we systematically use the following:

– We will always suppose that every (bordered) Riemann surface we deal with will have finite Euler characteristic and it is either compact or it is relatively compact in a bigger Riemann surface.

– *Mayer–Vietoris formula*: If F is a Riemann surface and U and V are two open sets of F then $\chi(F) = \chi(U) + \chi(V) - \chi(U \cap V)$.

– If β is a non compact Jordan curve which divides F in two connected components U and V then $\chi(F) = \chi(U) + \chi(V) + 1$. We will call β a *cross cut.*

– The Euler–Poincaré characteristic of a connected Riemann surface is at least -2 and it is -2 if and only if it is isomorphic to $\mathbb{P}^1$.

– Let $f : F \to G$ be a finite covering, Let U be a domain in G. A connected component V of $f^{-1}(U)$ is called a *island* if it is relatively compact in F and a *peninsula* otherwise.

Suppose that $P_1, \ldots, P_q$ are q points on $\mathbb{P}^1$ and $U_1, \ldots, U_q$ are small disks around the P_i's whose the closure are mutually disjoint. Denote by G^0 the Riemann surface $\mathbb{P}^1 \setminus \bigcup_i U_i$. We fix on $\mathbb{P}^1$ the Fubini–Study metric ω_{FS}: $A(\mathbb{P}^1; \omega_{FS}) = 1$.

If $f : F \to \mathbb{P}^1$ is a quasifinite morphism, then we denote by $N_i(f)$ the number of islands on F above U_i. The theorem which generalize the SMT is the following, it can be seen as a strong, non integrated form of it.

8.4 Theorem. *Suppose that we fixed* U_i *as above, then there is a positive constant* h *depending only on the* U_i*'s such that the following holds: for every Riemann surface* F *and quasifinite morphism* $f : F \to \mathbb{P}^1$ *we have that*

$$\chi^+(F) + \sum_i N_i(f) \geq (q-2)A(F; f^*(\omega_{FS})) - hL_f.$$

Theorem 8.4 is a consequence of 8.1 and 8.2. We give here a Sketch of the proof;

Sketch of Proof: Denote by G_0 the open set $\mathbb{P}^1 \setminus \bigcup_{i=1}^q \overline{U}_i$ and β the boundary of G_0. The Euler characteristic of G_0 is $q-2$. Denote by $\mathcal{I}$ (resp. $\mathcal{P}$) the set of islands (resp. peninsulas) of F over the U_i. Let F_0 be $f^{-1}(G_0)$ and $\gamma = f^{-1}(\beta)$. By Mayer Vietoris Formula, we have

$$\chi(F) = \chi(F_0) + \sum_{I \in \mathcal{I}} \chi(I) + \sum_{P \in \mathcal{P}} \chi(P) + n;$$

where n is the number of cross cuts of γ (components which are not compact). Since for every connected component A in the sum, $\chi(A) \geq -1$, each peninsula touch at least a cross cut and each cross cut touch at most one peninsula,

$$\chi^+(F) + \sum_i N_i(f) \geq \chi^+(G_0).$$

We conclude applying 8.1 and 8.2.

Denote by $n(f, P_i)$ the cardinality of the $z \in F$ such that $f(z) = P_i$ then one easily sees that $\sum_i n(f, P_i) \geq \sum_i N_i(f)$.

Let (Y, g) be a parabolic Riemann surface and $f : Y \to \mathbb{P}^1$ an analytic map. Apply the theorem to $F_t := \{z \in Y \text{ s.t. } g(z) \leq t\}$. It is well known that

$$\lim_{t\to\infty} \frac{\int_1^r \frac{L_{f_t} dt}{t}}{\int_1^r \frac{A(F_t, f^*(\omega_{FS}))dt}{t}} = 0;$$

where L_{f_t} is the length of the relative boundary of F_t (cf. for instance [BR]). Thus if one integrate the inequality of the theorem with respect to $\int_1^r \frac{dt}{t}$, one finds a proof of the SMT.

Remark that in the proof we are allowed to move a little bit the points P_i's and the results remains unchanged! This means that the SMT remains true if we perturb a little bit the divisor $D := \sum_i P_i$. This is the key point of the Yamanoi approach: In the theorem we can move a little bit the divisor and everything remains true, thus we can give a general proof working on the moduli space of stable pointed curves of genus zero, which is compact! The only problem is that sometimes the points P_i's may coincide.

One works directly on the moduli space of stable curves of genus zero with n marked points $\mathcal{M}_{0,n}$ and on its universal family $p : \mathcal{U}_{0,n} \to \mathcal{M}_{0,n}$. It is well known that there are n sections $\xi_i : \mathcal{M}_{0,n} \to \mathcal{U}_{0,n}$ of p and that $\mathcal{D}_n := \sum_i \xi_i(\mathcal{M}_{0,n})$ is the universal divisor: the restriction of $\mathcal{D}_n$ to the generic fibre of p is the divisor given by the marked points. Let $K_{\mathcal{U}/\mathcal{M}}$ be the relative dualizing sheaf of p. In the sequel we will denote by K_n the line bundle $K_{\mathcal{U}/\mathcal{M}}(\mathcal{D}_n)$ on $\mathcal{U}_{0,n}$; we will suppose that it is equipped with a smooth hermitian metric and we will denote by ω its first Chern form.

Since a rigorous proof is quite involved and requires a careful attention to details, we will explain the main steps of the proof in the case when $n = 4$ (stable curves of genus zero with 4 marked points). We refer to the original paper by Yamanoi for the general case. This is the first non trivial case which cannot be deduced directly from the classical SMT. Even in this case a detailed proof requires a skillful work (we think

that filling the gaps is a good exercise). Nevertheless we think that all the main steps and ideas of the proof are already present in this case.

8.5 *Explicit description of* $\mathcal{M}_{0,4}$ *and* $\mathcal{U}_{0,4}$. The moduli space $\mathcal{M}_{0,4}$ is isomorphic to the projective line $\mathbb{P}^1$.

Let $X := \mathbb{P}^1 \times \mathbb{P}^1$; we denote by $p : X \to \mathbb{P}^1$ the first projection. The map p is equipped with 4 sections: we write them in affine coordinate: $\xi_0(z) := (z, 0)$, $\xi_1(z) = (z, 1)$, $\xi_\infty(z) = (z, \infty)$ and $\xi_\Delta(z) = (z, z)$; we will denote by ξ_i and ξ_Δ the image of the ξ_i and of ξ_Δ respectively. The ξ_i, for $i = 0, 1, \infty$, do not intersect and ξ_Δ intersect the ξ_i properly over i.

Let $\pi : \tilde{X} \to X$ be the blow up of X over the three points $\xi_\Delta \cap \xi_i$. Then $\tilde{X}$ is the universal family $\mathcal{U}_{0,4}$ and the strict transforms $\hat{\xi}_j$'s of the ξ_j's, for $j = 0, 1, \infty$ and Δ are the universal sections. The map $p \circ \pi : \tilde{X} \to \mathbb{P}^1$ is the universal map.

Let $U_g \subset \mathbb{P}^1$ be the open set $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Then $\tilde{X}|_{U_g} := p^{-1}(U_g)$ is isomorphic to $U_g \times \mathbb{P}^1$; let $h : \tilde{X}|_{U_g} \to \mathbb{P}^1$ be the second projection. Suppose that $z = 0, 1$ or $z = \infty$, then we can find a neighborhood $U_x \subset \mathbb{P}^1$ of z for which $\tilde{X}|_{U_z} := p^{-1}(U_z)$ is the blow up of $U_z \times \mathbb{P}^1$ over the point (z, z); by construction there is a projection $h : (g_1; g_2) : \tilde{X}|_{U_x} \to \mathbb{P}^1 \times \mathbb{P}^1$. It is easy to see (by restriction to the fibres of p) that the restriction of K_4 to $\tilde{X}|_{U_g}$ is $h^*(\mathcal{O}(2))$. Moreover for $x = 0, 1, \infty$, (we may suppose that) the restriction of K_4 to $\tilde{X}|_{U_z}$ is $h^*(\mathcal{O}(1, 1))$. In the sequel we will suppose that the metric on the restriction of K_4 to these open sets is the pull back of the Fubini–Study metrics; this is not exactly the case but since $\tilde{X}$ is compact, the error we make is bounded and can be controlled.

In the sequel we will suppose that we are in the following situation: R will be a open set in $\mathbb{P}^1$ (for the analytic topology). $g : F \to R$ is a proper maps between Riemann surfaces and $f : F \to \tilde{X}$ is an analytic map such that the following diagram is commutative:

$$\begin{array}{ccc} F & \xrightarrow{f} & \tilde{X} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \pi} \\ R & \xrightarrow{\iota} & \mathbb{P}^1; \end{array}$$

where $\iota : R \to \mathbb{P}^1$ is the inclusion. We will call this *a situation*.

Suppose we are in a situation as above, and $W \subseteq F$ is a open set, we will denote by $A(W, \omega)$ the area of W with respect to the volume form $f^*(\omega)$ on F. If γ is a Jordan curve on F, we denote by $L(\gamma, \omega)$ the length of γ with respect to the measure defined by $f^*(\omega)$. We will denote by L_f the length of the relative boundary of f.

Let $D = \mathcal{D}_4 \hookrightarrow \tilde{X}$ be the universal divisor: we will denote by $n(D, f)$ the cardinality of the set $\{z \in F \ / \ f(z) \in D\}$.

9 The local version of the theorem.

The key step of Yamanoi proof is a local version of the theorem. This local version plays the role of theorem 8.4 in the Ahlfors proof of SMT. Given $f : F \to \tilde{X}$, and a open set U of $\tilde{X}$, we will generalize in the obvious way the concept of island and peninsula of F over U: an island will be a connected component of $f^{-1}(U)$ which is relatively compact, etc. We will denote by $N(f, U)$ the number of islands of F over U.

Before we state and prove the theorem, we need to state a generalization of a classical theorem by Rouché:

9.1 Proposition. *Let E be a Jordan domain of $\mathbb{P}^1$ and $b \in E$; then there exists a positive constant $C := C(E, b)$ with the following property: Let F be a bordered Riemann surface and $\zeta : F \to E$ an analytic function such that $\zeta(F) = E$ and $\zeta(\partial(F)) = \partial E$; then for every meromorphic function $\alpha : F \to \mathbb{P}^1$ such that $|\alpha(z) - b| < C$ for every $z \in F$, there exists $z \in F$ with $\alpha(z) = \zeta(z)$.*

The proof of this proposition is a variation of the classical Rouché theorem and can be found on [YA2].

The local version of Yamanoi theorem is

9.2 Theorem. *Let $x \in \mathbb{P}^1$ then we can find a open neighborhood $x \in U_x \subseteq \mathbb{P}^1$, open neighborhoods $W_i \subseteq \tilde{X}|_{U_x}$ of $\xi_i \cap \tilde{X}|_{U_x}$, for $i = 0, 1, \infty$ and Δ, with disjoint closures, and a positive constant h_x for which the following holds:*

For every situation

$$\begin{array}{ccc} F & \xrightarrow{f} & \tilde{X} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \pi} \\ R & \xrightarrow{\iota} & \mathbb{P}^1; \end{array}$$

for which $x \in R \subseteq U_x$, we have that

$$h_x L_f + \deg(g) + \chi^+(F) + \sum_i N(f, W_i) \geq A(F, \omega).$$

We recall that L_f is the length of the relative boundary. One sees the similarity of the theorem above with the classical theorem by Ahlfors 8.4. One should notice that 8.4 is one of the main tools of the proof of the theorem above.

Proof: First case: we suppose that $x \neq 0, 1$ or ∞. In this case the theorem is essentially 8.4; we give some details: Take a small disk U_x around x; then $\tilde{X}|_{U_x}$ is isomorphic to $U_x \times \mathbb{P}^1$; let $h : \tilde{X}|_{U_x} \to \mathbb{P}^1$ be the second projection. We may suppose that, for $i = 0, 1$ and ∞, we have $h \circ \xi_i(x) = i$ and $\xi_\Delta(x) = x$. Take small neighborhoods U_i of i in $\mathbb{P}^1$ and a small neighborhood U_Δ of x with non intersecting closures. We obtain the theorem in this case by applying Ahlfors theorem 8.4 to the morphism $h \circ f : F \to \mathbb{P}^1$. Notice that in this case the term $\deg(g)$ is not there.

The new case is when $x = 0, 1$ or ∞.

Suppose that $x = 0, 1$ or ∞: we may suppose that $x = 0$ the two other cases are similar.

In this case the fibre of $\pi : \tilde{X} \to \mathbb{P}^1$ over x is the union of two components X_1 and X_2 both isomorphic to $\mathbb{P}^1$. The universal sections ξ_0 and ξ_Δ intersect X_1 and not X_2 while ξ_1 and ξ_∞ intersect X_2 and not X_1. Take a neighborhood U_x of x and two maps $h_i : \tilde{X}|_{U_x} \to \mathbb{P}^1$. We may suppose that: $h_1(\xi_\infty(x)) = 0$, $h_1(\xi_1(x)) = 1$, $h_1(X_2) = \infty$ and that $h_2(\xi_0(x)) = 0$, $h_2(\xi_\Delta(x)) = 1$, $h_2(X_1) = \infty$.

For $i = 0, 1$ and ∞, choose small neighborhoods U_i of i whose closure do not intersect. Call $A := h_1^{-1}(U_\infty)$ and $B := h_2^{-1}(U_\infty)$. We may take the neighborhood U_x so small that $h_1(A \cap B) = h_2(A \cap B) = U_\infty$. We may also suppose that U_x is so small that $|h_j(\xi_i(z)) - \ell| < C(i, U_\ell)$ for $z \in U_x$, $j = 1$, $i = 0, 1, \Delta$ $\ell = 0, \infty, 1$ respectively, or $j = 2$ and $i = \ell = 0, 1, \infty$ respectively (cf. prop. 9.1).

For $i = 0, 1$ define the following open sets $W_i := h_1^{-1}(U_i)$ and $V_i := h_2^{-1}(U_i)$. Notice that the W_i and V_j are mutually disjoint and we may suppose U_x so small that $\xi_\infty(U_x)) \subset W_0$, $\xi_1(U_x)) \subset W_1$, $\xi_0(U_x) \subset V_0$ and $\xi_\Delta(U_x) \subset V_1$.

Let $A_1, \ldots, A_r$ be the island of F over $A \cap B$, $F_1 := F \setminus \{A_1, \ldots, A_r\}$ and $\mathcal{I}$ and $\mathcal{P}$ be the set of islands and peninsulas of F over B respectively. Remark that $\mathcal{I}$ is also the set of islands of F_1 over U_∞ via $h_1 \circ f$.

Let S be one of the A_i's or an element of one of $\mathcal{I}$. Prop. 9.1 applied to S, $h_1 \circ f|_S$ and $h_1 \circ \xi_1 \circ g$ implies that there is $z \in S$ such that $h_1 \circ f(z) = h_1 \circ \xi_1 \circ g(z)$. Since h_1 restricted to a fibre of π different from the fibre over x is an isomorphism, one finds that $\pi(f(z)) = x$. Consequently every such island intersects the fibre over x and the properness of g implies that $r + Card(\mathcal{I}) \leq \deg(g)$. In particular

$$\chi(F_1) + Card(\mathcal{I}) \leq \chi(F) + \deg(g). \tag{9.3.1}$$

Let N_A (resp. N_B) the number of islands of F_1 (or F which is the same) over V_0 and V_1(resp. over W_1 and W_2). A direct application of Ahlfors theory and Mayer–Vietoris formula to F_1 and $h_2 \circ f$ gives a universal constant h (independent on F) such that

$$\chi^+(F_1) + N_A - \sum_{P \in \mathcal{P}} \chi^+(P) - \sum_{I \in \mathcal{I}} \chi(I) \geq A(F; (h_2 \circ f)^*(\omega_{FS})) - hL_f. \tag{9.4.1}$$

Here and in the sequel, we systematically use theorem 8.1.

We apply again Ahlfors theory to each island and peninsula of F_1 over B. Observe that, for each island I, $\chi^+(I) < \chi(I) + 1$. Thus we obtain

$$\sum_{P \in \mathcal{P}} \chi^+(P) + \sum_{I \in \mathcal{I}} \chi(I) + Card(\mathcal{I}) + N_B \geq A(F; (h_1 \circ f)^*(\omega_{FS})) - hL_f. \tag{9.5.1}$$

The conclusion follows from 9.3.1, 9.4.1 and 9.5.1.

Remark that, since the base $\mathbb{P}^1$ is compact, the error we make using the pull back of the Fubini Study metric via h_i instead of the $(1,1)$ form ω of K_4 over $\tilde{X}$ is controlled by changing the constant h_x.

If we have a *situation* as above, we will denote by $n(D, f)$ the number of points $z \in F$ such that $f(z) \in D$. Let R_f be the number $\sum_{z \in F}(Ram(g) - 1)$

As a consequence, we find

9.5 Theorem. *Suppose that the we are in the hypotheses of theorem 9.2. Then*

$$A(F,\omega) \le n(D,f) + R_f + \deg(g)\chi(R) + \deg(g) + h_x L_F.$$

Proof: Since g is proper, $\chi^+(F) \le \chi(F) + \deg(g)$ and by Hurwitz formula, $\chi(F) = \deg(g)\chi(R) + R_f$. Thus it suffices to apply 9.2 and 9.1 to f and $\xi_i \circ g$ over each island.

Observe that the theorem above is a local version of the theorems; It seems better then the theorem because one has the impression that one can put $\epsilon = 0$; nevertheless there is the term coming from the relative boundary L_f. We will see in the sequel we will need to put $\epsilon > 0$ in order to control this term. Even if this is not the only reason, it is the most important.

10 The non integrated version of the theorem.

After the local version of the theorem we will prove a global non integrated version of the theorem. Here too we will put some restrictive hypotheses on the situations: nevertheless we would like to remark that these hypotheses are suffice to prove theorems 3.1 and 3.2.

Let K be a compact set of $\mathbb{P}^1$ (which may be empty) We will say that a sequence of open sets $R_n \subset \mathbb{P}^1$ is *relatively exhausting with respect to* K if, for every compact disk $\Delta \subset \mathbb{P}^1 \setminus K$, there exists n_0 such that, for every $n \ge n_0$ we have that $\Delta \subset R_n$.

The non integrated version of the theorems we propose is the following

10.1 Theorem. *Let K be a compact set of $\mathbb{P}^1$ and $\epsilon > 0$. Suppose that*

$$\begin{array}{ccc} F_n & \xrightarrow{f_n} & \tilde{X} \\ {\scriptstyle g_n}\downarrow & & \downarrow{\scriptstyle \pi} \\ R_n & \xrightarrow{\iota_n} & \mathbb{P}^1; \end{array}$$

is a sequence of situations with the sequence $\{R_n\}$ relatively exhausting with respect to K. Then, after subsequencing, we can find constants h and C such that, for every term of the subsequence

$$A(F_n;\omega) \le n(D,f_n) + R_{f_n} + \epsilon A(F_n,\omega) + h\ell(\partial F_n,\omega) + \deg(g_n)(\chi(R_n) + C).$$

The constant h is independent on the sequence and C depends only on the sequence (and not on the terms of the sequence).

Proof: We can find a open set W containing K having the following property: the open set $W \setminus K$ is a finite union of open sets of the form U_x of theorem 9.2 such that

$U_x \cap K \neq \emptyset$ and $x \notin K$. Choose an integer $J > \frac{2}{\epsilon}$. Let $\gamma_1, \ldots, \gamma_J$ be Jordan curves of $\mathbb{P}^1$ and δ_i small open neighborhoods of γ_i for which the following properties hold:
– For each i, every connected component of $\mathbb{P}^1 \setminus \gamma_i$ is simply connected and contained in one of the open sets U_x of theorem 9.2.
– For each i, every connected component of $\mathbb{P}^1 \setminus \delta_i$ is again simply connected (and contained in one of the U_x).
– For every triple of distinct indices (i, j, k) we have $\delta_i \cap \delta_j \cap \delta_k = \emptyset$.
– If a connected component of $\mathbb{P}^1 \setminus \delta_i$ intersects W then it is contained in it.

Because of the third condition, for every n, we have

$$\sum_j A(g_n^{-1}(\delta_j); \omega) \leq 2A(F_n, \omega).$$

Thus we can find a j_0 and a subsequence n_k for which

$$A(g_{n_k}^{-1}(\delta_{j_0}), \omega) \leq \frac{2}{J} A(F_{n_k}; \omega) \leq \epsilon A(F, \omega). \tag{10.2.1}$$

Fix such a j_0 and call δ the open set δ_{j_0} etc. From now on, we will omit to change notation when we pass to a subsequence.

Let U be a connected component of $\mathbb{P}^1 \setminus \delta$ and consider the set of Riemann surfaces $F_{n,U} := g_n^{-1}(R_n \cap U)$.

Either $\limsup_n \frac{A(F_{n,U},\omega)}{\deg(g_n)} < \infty$ or $\limsup_n \frac{A(F_{n,U},\omega)}{\deg(g_n)} = \infty$. We suppose that we are in the second situation, thus, passing to a subsequence, we may suppose that the lim sup is indeed a limit.

If U is not contained in W then we may also suppose that R_n contains U for every n. Suppose that we are in this case.

Denote by Δ_r the disk of radius r. We may suppose that U is biholomorphic to the disk Δ_{r_0} for some $r_0 < 1$. We may also suppose that $U \simeq \Delta_{r_0} \subset \Delta_1 \subsetneq U_x$ for some $x \in U$. Let $F_{n,\Delta} := g_n^{-1}(\Delta_1)$ and for every $r \in (0, 1)$ let $F_{n,r} := g_n^{-1}(\Delta_r) \subset F_{n,\Delta}$. We can find a non negative function G which is C^∞ outside the ramification points of g_n and integrable on $F_{n,\Delta}$, such that $f_n^*(\omega)|_{F_{n,\Delta}} = \sqrt{-1} G^2 dg_n \wedge d\overline{g}_n$. Let

$$S_n(r) := \int_0^r dt \int_{\partial F_{n,t}} Gt d\arg(g_n),$$

then $\frac{dS_n}{dr} = \ell(\partial F_{n,r}, \omega)$. By Cauchy Schwartz inequality we have

$$\begin{aligned} S_n(r) &\leq \left(\int_0^r td \wedge d\arg g_n \right)^{1/2} \cdot \left(\int_0^r G^2 t dt \wedge d\arg(g_n) \right)^{1/2} \\ &= C_U \cdot (\deg(g_n))^{1/2} \cdot (A(F_{n,r}; \omega))^{1/2}. \end{aligned}$$

Where $C_U > 0$ is a constant depending only on U.

10.2 Lemma. *Let $\delta > 0$ and $S_n(r)$ be a sequence of differentiable functions on* $[0, 1)$

with $S_n(r_0) \geq n^{2/\delta}$ then the set

$$I_S := \left\{ 1 > r \geq r_0 \ / \ S_n'(r) \geq \frac{S_n(r)}{(1-r^2)} \text{ for some } n \right\}$$

is such that $\int_{I_S} \frac{dr}{1-r^2} < \infty$.

The proof of the lemma is standard and can be found on [MQ2].

As a consequence of the lemma above, for every $\epsilon' > 0$ we can find a subsequence of the F_n and a $R > r_0$ for which $\ell(\partial F_{n,R}, \omega) < \epsilon' A(F_{n,R}, \omega)$. We call again U (by abuse of notations) the enlarged open set for which this last inequality holds.

A similar argument holds when U is contained in W. In this case, even taking a subsequence, we cannot suppose that U is contained in R_n: enlarging a bit U, as before, we may suppose

$$\ell(\partial F_{n,U}, \omega) \leq \epsilon A(F_{n,U}, \omega) + \ell(\partial F_n \cap F_{n,U}; \omega).$$

Since $F_n = \bigcup_U g_n^{-1}(U) \cap g_n^{-1}(\delta)$, and 10.2.1 holds, we apply theorem 9.5 and obtain

$$\begin{aligned}
&A(F_n; \omega) \leq \sum_U A(F_{n,U}, \omega) + A(g_n^{-1}(\delta)) \\
&\leq \sum_U \Big(n(D, f_n|_{F_{n,U}}) + R_{f_n|_{F_{n,U}}} + \deg(g_n)\chi(R_n \cap U) + \deg(g_n) + \epsilon A(F_{n,U}, \omega) \Big) \\
&+ \epsilon A(F_n, \omega) + h\ell(\partial F_n, \omega) + C,
\end{aligned}$$

Where the constant C take care of the open sets U for which $\limsup_n \frac{A(F_{n,U},\omega)}{\deg(g_n)} < \infty$ thus depends only on the sequence.

Since $\chi(R_n) \geq \sum_U \chi(R_n \cap U)$ we conclude that

$$A(F_n; \omega) \leq n(D, f_n) + R_{f_n} + \epsilon A(F_n, \omega) + h\ell(\partial F_n, \omega) + \deg(g_n)(\chi(R_n) + C).$$

From this we deduce

10.3 Theorem. *Let $\epsilon > 0$ then there exists constants C and h such that,* for every *situation as above,*

$$A(F; \omega) \leq n(D, f) + R_f + \epsilon A(F, \omega) + h\ell(\partial F, \omega) + \deg(g)(\chi(R) + C).$$

Proof: If not, we can find a sequence of situations for which

$$\lim_{n \to \infty} \frac{1}{\deg(g_n)} \cdot ((1-\epsilon)A(F_n; \omega) - (n(D, f_n) + R_{f_n} + h + h\ell(\partial F_n, \omega))) + \chi(R_n) = +\infty.$$

And this contradicts theorem 10.1.

This easily imply, together with 7.1 the algebraic version of *abc*. The analytic version of *abc* requires again a control of the length of the boundary; this is again standard: We

give a sketch of the proof in a special case. We suppose, to simplify that $g : Y \to \mathbb{C}$ is a proper map and the following diagram is commutative

$$\begin{array}{ccc} Y & \xrightarrow{f} & \tilde{X} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbb{C} & \xrightarrow{\iota} & \mathbb{P}^1. \end{array}$$

Applying theorem 10.3 when $R = R_t$, the disk of radius t in $\mathbb{C}$, and integrating with respect to $\int_0^r \frac{dt}{t}$, we obtain

$$(K_4, Y)(r) \leq N^{(1)}_{D_4}(Y)(r) + R_f(r) + \epsilon(K_4, Y)(r) + h \int_0^r \frac{\ell(\partial g^{-1}(R_t), \omega) dt}{t} + C \log r.$$

We can write $f^*(\omega) = \sqrt{-1} G^2 dg \wedge d\overline{g}$ with G a non negative function which is integrable and C^∞ outside the ramification points of g.

Introduce the function $S(r) := \int_0^{\log(r)} dt \int_{g=t} tGd\arg(g)$. We have that $S'(r) = \int_0^r \frac{\ell(\partial g^{-1}(R_t),\omega)dt}{t}$ and Cauchy–Schwartz inequality gives $S(r) \leq C(\log(r))^{1/2} \cdot \frac{d(K_4;Y)(r)}{dr}$. A double application of lemma 4.4 allows to conclude the proof. Remark that this argument is similar to the argument used to derive the SMT from Ahlfors theory.

10.4 *The general statement proved by Yamanoi.* As a conclusion, we state without proof the main theorem proved in [YA3]; we refer to the original paper for the proof. Let $n > 3$ be an integer. A *situation* will be a commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{f} & \mathcal{U}_{0,n} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \pi} \\ R & \xrightarrow{\iota} & \mathcal{M}_{0,n}; \end{array}$$

with F and R bordered Riemann surfaces, g a proper analytic map and f and ι analytic. We fix a metric on K_n and a positive $(1,1)$ form η on $\mathcal{M}_{0,n}$. We define $n(\mathcal{D}_n, f)$ and R_g as before. Observe that $\mathcal{M}_{0,n} = \mathcal{U}_{0,n-1}$ thus we may define $n(\mathcal{D}_{n-1}, \iota)$.

10.4 Theorem. *Let $\epsilon > 0$ then there is a constant C depending only on ϵ and the metrics chosen on K_n and $\mathcal{M}_{0,n}$ with the following property: For every situation as above, we have*

$$\begin{aligned} A(F, f^*(c_1(K_n))) &\leq n(\mathcal{D}_n, f) + R_g + \epsilon A(F, f^*(c_1(K_n))) \\ &\quad + C \deg(g) \left(A(R, g^*(\eta)) + n(\mathcal{D}_{n-1}, \iota) + \chi^+(R) + \ell(\partial F, f^*(c_1(K_n))) \right). \end{aligned}$$

Cf. [YA3] Theorem 4. An argument similar to the one sketched above allows to deduce the *abc* conjecture from the theorem above.

11 Conclusions and final observations.

A posteriori one would like to compare the two proofs. The proof by McQuillan has a global nature while the Yamanoi 's is more local. Of course one is tempted to apply the techniques to other situations; for instance to families of surfaces of general type. The first part of the proof by McQuillan passes through without pain (essentially everything until prop. 6.5). Then one have to deal with a more subtle situation: here we strongly used the fact that the singularities of families of semistable curves are well understood and quite easy. In general the situation is more complicated.

The Yamanoi approach is essentially local. Suppose that we have a family of varieties over a curve and we want an inequality similar to the *abc* in this situation. Split the base in finitely many small open sets U_i. Take a sequence of curves with maps in our family. Look to the sequence of the areas of the preimages of each U_i. If this is bounded, there is nothing to prove. If it is unbounded, then one look for a local inequality which will involve the length of the boundary as in theorem 9.5. Then one can conclude adapting the arguments of theorem 10.1. Of course this will need a generalization of Ahlfors theory (even in the smooth case) and to our knowledge this is still unknown.

In conclusion, the first part of the proof by McQuillan and the second part of the proof by Yamanoi can be generalized. Thus it is probable that the best way to proceed will be by applying a mix of both proofs!

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